SIAM J. MATH. ANAL.
Vol. 38, No. 5, pp. 1565–1587
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EXPONENTIAL HOMOGENIZATION OF LINEAR SECOND ORDER ELLIPTIC PDEs WITH PERIODIC COEFFICIENTS*

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Abstract. A problem of homogenization of a divergence-type second order uniformly elliptic operator is considered with arbitrary bounded rapidly oscillating periodic coefficients, either with periodic "outer" boundary conditions or in the whole space. It is proved that if the right-hand side is Gevrey regular (in particular, analytic), then by optimally truncating the full two-scale asymptotic expansion for the solution one obtains an approximation with an exponentially small error. The optimality of the exponential error bound is established for a one-dimensional example by proving the analogous lower bound.

Key words. homogenization, elliptic equation, exponential asymptotics, Gevrey regularity, and analyticity

AMS subject classifications. 35B27, 35C20, 35J25, 74Q05

DOI. 10.1137/060651045

1. Introduction.

Classical homogenization theory describes the relation of solutions $u^\varepsilon(x)$ of boundary value problems with rapidly oscillating coefficients to solutions $u_0(x)$ of a homogenized problem, i.e., a problem without rapidly oscillating coefficients. In appropriate function spaces convergence can be established as $\varepsilon \to 0$ (with ε describing the period or wavelength of the coefficients' oscillations); see, e.g., [1, 2, 3, 4] and the references therein. For particular homogenization problems, e.g., for those described by linear second order elliptic PDEs with periodic coefficients, the rate of convergence with respect to ε can often also be determined, see, e.g., [4, 5, 6, 7]. The order of convergence can sometimes be improved further by constructing higher order correctors. The presence of a boundary creates additional "boundary layers," which substantially complexifies the problem of constructing the higher order terms; see, e.g., [5, 3, 4, 8, 9]. However, in the absence of the boundary, either for a problem with outer periodicity conditions or in the whole space (away from the spectrum), higher order terms can often be explicitly constructed. In particular, under the assumptions of sufficient regularity of the coefficients and the right-hand side of the equation, it is possible to construct and rigorously justify a full two-scale asymptotic expansion for $u^\varepsilon(x)$, i.e., to establish the error bounds both for linear problems (e.g., [3, 10]) and even for appropriate nonlinear ones [11].

The above can be referred to, in the context of homogenization, as "homogenization in all orders," by analogy with "asymptotics in all orders": by appropriately

*Received by the editors January 27, 2006; accepted for publication (in revised form) August 21, 2006; published electronically January 12, 2007. This work was partly supported by Bath Institute for Complex Systems (EPSRC grant GR/S86525/01) and by the Institute for Mathematics and Its Applications (IMA), University of Minnesota. Part of this work was done during the authors' visits to IMA during February–June 2005.
http://www.siam.org/journals/sima/38-5/65104.html
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truncating the infinite asymptotic series one arrives at an asymptotic approximation to the actual solution u^ε with accuracy of any desirable polynomial order in ε as $\varepsilon \to 0$. We address in this paper the question of homogenization "beyond all orders," i.e., with an exponentially small error, via an optimal truncation of the (generally divergent) asymptotic expansion. The ideas of exponential asymptotics have been intensively developed in the recent literature (see, e.g., [12] and the references therein); however, not that much progress has been achieved in this direction specifically for problems of averaging and more specifically of homogenization. An exponential averaging technique was developed for ODEs by Neishtadt [13] and recently adjusted to PDEs with a temporal [14] and then *one-dimensional spatial* oscillations [15].

To the best of our knowledge, the present work represents the first example of a rigorous analytic exponential averaging for truly *multidimensional spatial oscillations*, i.e., for multidimensional homogenization. On the other hand, exponentially accurate approximations are potentially relevant to the problem of achieving exponentially convergent numerical schemes for homogenization; see, e.g., [16].

We consider the abovementioned "classical" elliptic homogenization problems with periodic coefficients, both for the case of periodic boundary conditions and in the whole space. We will assume that the right-hand side is sufficiently regular (not only infinitely smooth as required for constructing the full asymptotic expansion, but additionally "Gevrey regular," in particular, analytic). Then we show that one obtains an approximation with an exponentially small error by optimally truncating the full two-scale asymptotic series for the solution. Importantly, the above exponential bounds are sharp in the sense that we establish analogous lower bounds for the error in an explicit but rather generic one-dimensional example.

The Gevrey regularity techniques have proved useful in exploring exponentially small effects in different problems, for example, in diffraction/scattering for describing the wave field in the shadow [17] and the asymptotic distribution of resonances [18], and in the one-dimensional exponential averaging [14, 15] for controlling the effect of Galerkin approximation of PDEs via ODEs. In the present work, however, the Gevrey regularity allows us to control the error of the truncation of a full asymptotic expansion both with respect to the short period or wavelength of the oscillations ε and the large number n of the terms in the truncated asymptotic series.

The next section gives a precise formulation of the problems and the statements of the main results, which are Theorems 1 and 1′, and specifically the exponential error bounds (17) and (19). The rest of the paper is devoted to the proof of the theorems, as well as of the optimality of the estimates (17) and (19) for an explicit one-dimensional example; see Theorem 5. In particular, for analytic right-hand sides $f(x)$, in Theorems 1 and 5, the exponential bounds (17) and (79) hold with $\beta = 1$, with the "rate" of decay (the constants C_2 and $\tilde{C}_2$) related to the imaginary part of the "nearest" singularity in the analytic continuation of $f(x)$ for complex x; see Remark 7.

2. Statement of the problem and main results. We consider a family of differential operators with rapidly oscillating periodic coefficients:

$$(1) \qquad (L^\varepsilon u)(x) := -\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right) \nabla u \right)(x).$$

The matrix $A(y) = (A_{ij}(y))_{ij} \in L^\infty(\mathbb{T})$, $i, j = 1, \dots, d$, where $\mathbb{T} = \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 1$, is a d-dimensional torus, is assumed to be symmetric[1] and uniformly elliptic; i.e.,

[1]The assumption of the symmetry of matrix $A(y)$ holds in most physically relevant examples, but could be waived for the purposes of this paper: the stated results would still hold at the expense of a slightly more complicated algebra in the exposition.

$A_{ij}(y) = A_{ji}(y)$ for any i, j and $y \in \mathbb{T}$ and there exists $\nu_0 > 0$ such that for all $\xi \in \mathbb{R}^d$ and $y \in \mathbb{T}$

$$A_{ij}(y)\xi_i\xi_j \geq \nu_0|\xi|^2. \tag{2}$$

Here and throughout the paper we use the Einstein summation convention with respect to repeated indices.

The main problem considered in this paper is for the right-hand side f being infinitely smooth and periodic with a "fixed" period chosen to be equal to unity and having zero mean, with the solution also required to have zero mean and to satisfy the periodic boundary conditions; cf. [3, 10]. Namely, assuming $\varepsilon^{-1} \in \mathbb{N}$ to be a large integer, we address the following homogenization problem: for a given f with zero-mean value

$$\langle f \rangle := \int_{\mathbb{T}} f(x)\, dx = 0, \tag{3}$$

we seek a solution to the problem

$$(L^\varepsilon u^\varepsilon)(x) = f(x) \text{ in } \mathbb{T}, \tag{4}$$

$$\langle u^\varepsilon \rangle := \int_{\mathbb{T}} u^\varepsilon(x)\, dx = 0. \tag{5}$$

Equation (4) is a "classical" model of periodic homogenization, physically corresponding to, e.g., stationary heat conduction, electric conductivity, linear elasticity in anti-plane shear, etc.

For a special class of functions f, namely for Gevrey regular functions, we will construct an exponentially accurate asymptotic approximation to u^ε. Thus, we adopt the following definition (cf., e.g., [20], [21]).

Definition 1. *We say that a $C^\infty(\mathbb{T})$ function f is β-Gevrey regular, where $\beta \geq 1$, if there exists $B > 0$ such that for all $l \in \mathbb{N}$*

$$\|f\ ;\ H^l(\mathbb{T})\| \leq B^l (l!)^\beta, \tag{6}$$

where B may depend on f but is independent of l. We use notation $f \in \mathcal{G}^\beta(\mathbb{T})$.

Here and below we use the scale of Sobolev spaces $H^l(\mathbb{X})$, $l \in \mathbb{N}$, on a Riemannian manifold $\mathbb{X}$, with the norm

$$\|f\ ;\ H^l(\mathbb{X})\| = \sum_{|k|=l} \|D^k f\ ;\ L^2(\mathbb{X})\| + \|f\ ;\ L^2(\mathbb{X})\|, \tag{7}$$

where $\|\cdot\ ;\ L^2(\mathbb{X})\|$ is the standard L^2 norm on $\mathbb{X}$, and we adopt the following conventional multi-index notation: $k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$, where $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ is the set of nonnegative integers; $|k| := k_1 + \cdots + k_d$; and $D^k := \partial^{|k|}/\partial x_1^{k_1} \ldots \partial x_d^{k_d}$. We will also deal with $H^{-1}(\mathbb{X})$ norms, defined as duals to the space $H_0^1(\mathbb{X})$ of functions from $H^1(\mathbb{X})$ with zero mean.

Definition 1 gives one of several equivalent definitions of the Gevrey "extreme regularity" class $\mathcal{G}^\beta$ (see also [19]). In particular, for $\beta = 1$ the functions are from $\mathcal{G}^\beta$ if and only if they are real analytic; for $\beta > 1$ the functions are infinitely smooth but not necessarily analytic. A conventional way of clarifying these relations is by reformulating them in the Fourier space. For the above $\mathbb{T}$-periodic functions f, when represented by their Fourier series

$$f(x) = \sum_{p \in \mathbb{Z}^d} f_p \exp(2i\pi p \cdot x), \tag{8}$$

a sufficient condition for f to belong to $\mathcal{G}^\beta$ is for its Fourier coefficients f_p to decay exponentially with the "rate" $|p|^{1/\beta}$; i.e.,

$$|f_p| \le c_1 \exp\Big(-c_2|p|^{1/\beta}\Big) \tag{9}$$

with some p-independent positive constants c_1 and c_2. The latter is well known and can be seen, for example, by applying the Plancherel theorem to (7), using (9), then replacing the resulting series by "asymptotically equivalent" integrals, and finally employing the Stirling asymptotic formulae; see, e.g., [29, (6.1.37)]. Throughout the paper we will use various minor modifications of the direct implication of the Stirling formula for

$$\Gamma(z) := \int_0^\infty \exp(-s)s^{z-1}ds, \ z > 0; \ \Gamma(l+1) = l!, \ l \in \mathbb{N}, \tag{10}$$

which we display below for the reader's convenience:

$$A_1\left(\frac{z}{e}\right)^{z-1/2} \le \Gamma(z) \le A_2\left(\frac{z}{e}\right)^{z-1/2}, \quad z \ge 1, \tag{11}$$

with some "universal" constants A_1 and A_2.

Notice that for real-analytic f (9) holds with $\beta = 1$, and the rate of exponential decay c_2 is determined by the absolute value of the imaginary part of the "nearest" singularity in the analytic continuation.

For any fixed $\varepsilon > 0$ the problem (4)–(5) is a well-posed elliptic problem which has a unique solution $u^\varepsilon \in H^1(\mathbb{T})$. Given $n \in \mathbb{Z}_+$ we seek an approximation to this solution in the standard form of the appropriately truncated two-scale asymptotic series (cf., e.g., [3]):

$$u^{\varepsilon,n}(x) = \sum_{m=0}^{n+2} \varepsilon^m u^{(m)}\left(x, \frac{x}{\varepsilon}\right), \tag{12}$$

where the functions $u^{(m)}(x,y)$ are required to be periodic in the "fast" variable y. It is known that for the present problem one can construct in this way a full asymptotic expansion with $u^{(m)}$ adopting the following form (see, e.g., [3, 10]):

$$u^{(m)}(x,y) = \sum_{l=0}^{m} \sum_{|k|=l} N_k(y) D_x^k v_{m-l}(x), \tag{13}$$

where $N_0(y) \equiv 1$ and $N_k(y)$ are periodic solutions of the "main" ($|k| = 1$) and "higher order" ($|k| > 1$) "canonical" unit cell problems in the "fast" variable y. The functions $v_s(x)$, $s \ge 0$, solve certain recurrent systems of equations in the "slow" variable x (see [3]), which are briefly reviewed in the next section.

Before formulating the main result, for convenience of the future referencing, we combine (12) and (13) to give

$$u^{\varepsilon,n}(x) = \sum_{l=0}^{n+2} \varepsilon^l \sum_{|k|=l} N_k\left(\frac{x}{\varepsilon}\right) D_x^k V^{(n-l+2)}(x,\varepsilon). \tag{14}$$

The slowly varying part in (14) is a partial sum of the formal asymptotic series $V^{(\infty)}(x,\varepsilon)$ (see (29)):

$$V^{(M)}(x,\varepsilon) := \sum_{s=0}^{M} \varepsilon^s v_s(x). \tag{15}$$

The main results of the present paper are the following.

THEOREM 1. *Suppose* $A \in L^\infty(\mathbb{T})$ *and satisfies* (2), *and* $f \in \mathcal{G}^\beta(\mathbb{T}), \beta \geq 1, \langle f \rangle = 0$. *Let* u^ε *be the unique solution of* (4)–(5). *Then there exist* ε*-independent constants* $C_1 > 0$, $C_2 > 0$, $\kappa_1 > 0$, *and* $\kappa_2 > \kappa_1$, *such that for any* n *satisfying*

$$\kappa_1 \varepsilon^{-1/\beta} \leq n \leq \kappa_2 \varepsilon^{-1/\beta} \tag{16}$$

the approximation (14) *has the error bound*

$$\| u^\varepsilon - u^{\varepsilon,n} \ ; \ H^1(\mathbb{T}) \| \leq C_1 \exp(-C_2 \varepsilon^{-\frac{1}{\beta}}). \tag{17}$$

The above result may be interpreted in the sense that if the (generally divergent) asymptotic series (12) is, for sufficiently small ε, "optimally" truncated by choosing $n = n(\varepsilon)$ according to (16), for example, $n(\varepsilon) = \left[\kappa_2 \varepsilon^{-1/\beta}\right]$ with the square brackets denoting the entire part, then this produces an exponentially small error in the sense of (17).

Note also that for less regular f the earlier results on the polynomial rather than exponential error (see, e.g., [5, Thm. 11.1], [3, section 4.2, Thm. 2]) will be a by-product of our analysis: if, e.g., f has a finite regularity in the scale of Sobolev spaces, say f belongs to $H^M(\mathbb{T})$ but does not belong to $H^{M+1}(\mathbb{T})$ for some M, one can construct only finitely many terms in the expansion (12). As a result one obtains only an error bound of polynomial order ε^n with a finite n related to M. On the other hand, if one assumes $f \in C^\infty$ but makes no assumption on the "rate" of growth of its H^l norms when $l \to \infty$, one does reproduce the "homogenization in all orders" with an error bound $C_n \varepsilon^n$ for any n. However, in the latter case one has no control on the growth of C_n as $n \to \infty$, which disallows any possible further "a priori" improvement of the error bound.

Let us also note that Theorem 1 can be generalized further in a number of ways. Assuming higher regularity of the coefficients A_{ij}, one can get in (17) the same rate of convergence but in stronger norms. One can also consider another case without the boundary for a "shifted" operator $L^\varepsilon + 1$ in entire $\mathbb{R}^d$ rather than for L^ε in a fixed domain with periodic boundary condition. Then the same exponential estimate holds; i.e., the following theorem can be obtained adapting the proof of Theorem 1 with minor changes.

THEOREM 1′. *Suppose* $A \in L^\infty(\mathbb{T})$ *and satisfies* (2); $f \in \mathcal{G}^\beta(\mathbb{R}^d), \beta \geq 1$, *i.e.,* $f \in C^\infty(\mathbb{R}^d)$ *and there exists* $B > 0$ *such that for all* $l \in \mathbb{N}, \| f \ ; \ H^l(\mathbb{R}^d) \| \leq B^l (l!)^\beta$. *Let* $u^\varepsilon \in H^1(\mathbb{R}^d)$ *be the unique solution of*

$$(L^\varepsilon + 1) u^\varepsilon = f. \tag{18}$$

Then there exist ε*-independent constants* $C_1 > 0$, $C_2 > 0$, $\kappa_1 > 0$, *and* $\kappa_2 > \kappa_1$, *such that for any* n *satisfying* $\kappa_1 \varepsilon^{-1/\beta} \leq n \leq \kappa_2 \varepsilon^{-1/\beta}$ *the corresponding asymptotic approximation of the form* (12) *has the error bound*

$$\| u^\varepsilon - u^{\varepsilon,n} \ ; \ H^1(\mathbb{R}^d) \| \leq C_1 \exp(-C_2 \varepsilon^{-\frac{1}{\beta}}). \tag{19}$$

Note that in the latter case the explicit structure of the two-scale asymptotics (12) is slightly different from that of (14); see (57).

We expect similar results to be valid also for *nonlinear* elliptic divergence operators (cf. [11]). Accounting for the presence of a boundary is in general a difficult open problem; cf. [8, 9].

The proof of the theorems will be divided into three steps. First we derive a priori estimates on appropriate norms of the coefficients N_k and v_s in suitable functional spaces for fixed k and s in section 3. Then, in section 4, we estimate the right-hand side error term $L^\varepsilon u^{\varepsilon,n} - f$ for fixed n and ε (in the H^{-1} norm). In section 5 we translate this into the error estimates for $u^{\varepsilon,n(\varepsilon)} - u^\varepsilon$ via analysis of the "mean" and standard ellipticity estimates, and finally "minimize" the error by an optimal choice of $n(\varepsilon)$ dependence on ε. This establishes the desired exponential error bound and hence proves Theorem 1. Proof of Theorem $1'$ follows the same strategy with minor technical alterations listed in Remark 2 immediately following the proof of Theorem 1.

Optimality of the exponential error bound (17) is proved in section 6 for an explicit one-dimensional example; see Theorem 5.

3. Recurrent relations and a priori estimates. We briefly describe the procedure for determining the coefficients in (14) and (15) (see, e.g., [3, 10] for the detailed derivation in a slightly different notation). Below, we give one possible way to summarize it.

First, the infinite series version of (14),

$$u^\varepsilon(x) \sim \sum_{l=0}^{\infty} \varepsilon^l \sum_{|k|=l} N_k\left(\frac{x}{\varepsilon}\right) D_x^k V^\infty(x,\varepsilon), \tag{20}$$

is formally substituted into (4). After appropriate differentiations and re-grouping the terms with equal powers of ε (treating at this stage $V^\infty(x,\varepsilon)$ as a "whole") we arrive at

$$\sum_{l=0}^{\infty} \varepsilon^{l-2} \sum_{|k|=l} \left\{L_y^1 N_k(y) - T_k(y)\right\}_{y=x/\varepsilon} D^k V^\infty(x,\varepsilon) \sim f(x), \tag{21}$$

where

$$N_0(y) \equiv 1, \quad T_0(y) \equiv 0, \tag{22}$$

and

$$|k| = 1 : T_k(y) = A_{ij,j}(y), \; k = e_i, \tag{23}$$

$$|k| \geq 2 : T_k(y) = \sum_{\substack{i,j=1,\dots,d \\ k'=k-e_i\geq 0}} \left((A_{ij}N_{k'})_{,j} + A_{ij}N_{k',j}\right)(y) + \sum_{\substack{i,j=1,\dots,d \\ k''=k-e_i-e_j\geq 0}} A_{ij}(y)N_{k''}(y). \tag{24}$$

Here we denote by e_i the unit ith axis vector in $\mathbb{Z}^d$ and adopt the standard convention denoting derivatives by the indices following the comma in the subscript; $k \geq 0$ for a multi-index k means $k_i \geq 0$ for any $1 \leq i \leq d$ ($k > 0$ will mean $k \geq 0$ and $k_i > 0$ for some i, with $k' < k$ meaning $k - k' > 0$, etc.).

We then require the "coefficients" $\left\{L_y^1 N_k(y) - T_k(y)\right\}$ in (21) to be independent of the fast variable y, i.e., to be equal to constants which are denoted by $-h_k$. This implies that N_k are solutions to the following "cell problems" for $|k| \geq 1$:

$$L_y^1 N_k = T_k(y) - h_k \text{ in } \mathbb{T}, \tag{25}$$

$$\langle N_k \rangle = 0, \tag{26}$$

with periodic conditions for N_k.

The solvability condition for (25)–(26) implies, necessarily, that h_k are the mean values of T_k over the periodicity cell:

$$h_k = \langle T_k \rangle. \tag{27}$$

Combining (21) with (25) yields an infinite order formal asymptotic equation for the "slow" part $V^\infty(x, \varepsilon)$:

$$-\sum_{l=0}^{\infty} \varepsilon^{l-2} \sum_{|k|=l} h_k D^k V^\infty(x, \varepsilon) \sim f(x). \tag{28}$$

A formal asymptotic solution of (28) is in turn sought in the form of an "infinite order" version of (15):

$$V^\infty(x, \varepsilon) \sim \sum_{s=0}^{\infty} \varepsilon^s v_s(x). \tag{29}$$

The substitution of (29) into (28) with subsequent rearrangements and equating terms with the same powers of ε yields

$$-A^{\text{hom}}_{i,j=1} v_{s,ij}(x) = f_s(x), \tag{30}$$

$$\langle v_s \rangle = 0, \tag{31}$$

with the right-hand sides

$$f_0 = f, \tag{32}$$

$$f_s = \sum_{l=3}^{s+2} \sum_{|k|=l} h_k D_x^k v_{s-l+2}, \;\; s \geq 1. \tag{33}$$

In (30) $A^{\text{hom}} = \left(A^{\text{hom}}_{ij}\right)_{i,j=1}^{d}$ is a "classical" homogenized matrix, which is known to be positive definite (with the same ellipticity constant ν_0 as in (2)) and symmetric:

$$A^{\text{hom}}_{ij} = \langle A_{ij} \rangle + \langle A_{is} N_{e_j,s} \rangle = \begin{cases} h_{e_i+e_j}, & i = j, \\ \frac{1}{2} h_{e_i+e_j}, & i \neq j. \end{cases}$$

Notice that (30) is uniquely solvable for any $s \geq 0$: a necessary and sufficient condition for the solvability is $\langle f_s \rangle = 0$ which does hold for $s = 0$ by assumption (3) and for $s \geq 1$ by (33). The slowly varying terms v_s are hence found recurrently as solutions to *homogenized* equations (30) with constant coefficients on a torus $\mathbb{T}$.

The relations (22)–(27) and (30)–(33) are hence sufficient for uniquely identifying all N_k and v_s, respectively. Then the so-defined asymptotic "double series" (20), (29) provides a full asymptotic expansion of the solution $u^\varepsilon(x)$ "in all orders": in particular, its truncation $u^{\varepsilon,n}$ produces an error of polynomial order in ε (see, e.g., [3, section 4.2, Thm. 2] or section 4 below).

We next aim at estimating the quantities

$$\mathcal{N}^{(l)} := \max_{|k|=l} \|N_k \ ; \ H^1(\mathbb{T})\|, \tag{34}$$

$$\mathcal{V}^{(s,m)} := \|v_s \ ; \ H^m(\mathbb{T})\|. \tag{35}$$

We will prove the following lemma.

LEMMA 2. *Under the assumptions of Theorem* 1 *the following estimates hold for all* $l, s, m \in \mathbb{N} \cup \{0\}$: (i)

$$\mathcal{N}^{(l)} \le (M_{\mathcal{N}})^l, \tag{36}$$

(ii)

$$\mathcal{V}^{(s,m)} \le \sum_{k=0}^{s} (M_{\mathcal{V}})^{k+1} \|f \ ; \ H^{m+k}(\mathbb{T})\| \tag{37}$$

for appropriate constants $M_{\mathcal{N}}$ *and* $M_{\mathcal{V}}$, *depending only on* $\|A \ ; \ L^\infty(\mathbb{T})\|$ *and the ellipticity constant* ν_0 *(see* (2)*).*

Proof. Further on we will use the abbreviated notation $|\cdot|_l := \|\cdot \ ; \ H^l(\mathbb{T})\|$, $l \ge -1$ ($|\cdot|_0 := \|\cdot \ ; \ L^2(\mathbb{T})\|$) and denote by C, M_1, M_2, etc. various positive constants whose precise values are insignificant and can change during the proof.

(i) Due to the standard ellipticity estimates we have

$$|v|_1 \le C(\nu_0)|G|_0 \tag{38}$$

for a solution of $L^1 v = \nabla \cdot G$, $\langle v \rangle = 0$ with arbitrary $G \in \left(L^2(\mathbb{T})\right)^d$. So we deduce from (22)–(24), (25)–(26), and (27) for $|k| \ge 2$ that

$$|N_k|_1 \le C(\nu_0)\|A \ ; L^\infty(\mathbb{T})\| \left(\sum_{k'<k,\, |k-k'|=1} |N_{k'}|_1 + \sum_{k''<k, |k-k''|=2} |N_{k''}|_0 \right). \tag{39}$$

The latter reads in terms of (34) as

$$\mathcal{N}^{(l)} \le M_1 \mathcal{N}^{(l-1)} + M_2 \mathcal{N}^{(l-2)}, \quad l \ge 2, \tag{40}$$

and implies (36) by induction: from (22) we have $\mathcal{N}^{(0)} = 1$ and, due to (23) and (38), $\mathcal{N}^{(1)} \le C(\nu_0)\|A \ ; L^\infty(\mathbb{T})\| \le M_1$. Therefore choosing $M_{\mathcal{N}} > \max\{1, M_1 + (M_2)^{1/2}\}$ we arrive at (36).

(ii) Now turning to v_s, due to (30)–(33) we estimate for $s \ge 1$ and $m \ge 1$

$$|v_s|_m \le C(A^{\mathrm{hom}}) \sum_{l=3}^{s+2} \sum_{|k|=l} |h_k| |D_x^k v_{s-l+2}|_{m-2}. \tag{41}$$

This can be established, e.g., using again the ellipticity estimates applied to (30), which being an elliptic equation with constant coefficients can be differentiated m times. Applying also a version of the Poincaré inequality, which in our choice of the domain and the norms (see (7)) is the obvious estimate

$$\|g; H^k(\mathbb{T})\| \le \|g; H^l(\mathbb{T})\|, \ k \le l, \tag{42}$$

we conclude that $C(A^{\mathrm{hom}})$ can be chosen independently of m.

Since (24) and (27) obviously imply that $\max_{|k|=l} |h_k| \leq C\left(\mathcal{N}^{(l-1)} + \mathcal{N}^{(l-2)}\right)$, from (41) we arrive after a straightforward manipulation at

$$\mathcal{V}^{(s,m)} \leq C \sum_{r=1}^{s} \mathcal{N}^{(r+1)} r^{d-1} \mathcal{V}^{(s-r,m+r)}. \tag{43}$$

The latter in turn, combined with (36), implies that with large enough M_0

$$\mathcal{V}^{(s,m)} \leq \sum_{r=1}^{s} M_0^r \mathcal{V}^{(s-r,m+r)}. \tag{44}$$

Let us finally show by induction in s that the latter is sufficient to deduce (37) with some $M_{\mathcal{V}} > 2M_0$. Indeed for $s = 0$, due to (30) for all $m \geq 0$, we have $|v_0|_m \leq |v_0|_{m+2} \leq M_3|f|_m$ with M_3 independent of m, implying (37). Now we proceed with the induction step: suppose (37) holds for $s = 0, \ldots, S$ with a constant $M_{\mathcal{V}} > \max\{2M_0, M_3\}$. Then due to (44) we have

$$\begin{aligned}\mathcal{V}^{(S+1,m)} &\leq \sum_{r=1}^{S+1} M_0^r \mathcal{V}^{(S+1-r,m+r)} \leq \sum_{r=1}^{S+1} \sum_{k=0}^{S+1-r} M_0^r M_{\mathcal{V}}^{k+1} |f|_{m+r+k} \\ &\leq \sum_{q=1}^{S+1} M_{\mathcal{V}}^{q+1} |f|_{m+q} \sum_{r=1}^{q} \left(\frac{M_0}{M_{\mathcal{V}}}\right)^r,\end{aligned}$$

which by our choice of $M_{\mathcal{V}}$ implies (37) for $s = S+1$. ☐

4. Remainder estimates. Next we derive estimates for the error in the right-hand side of the original equation (4) as a result of substitution into its left-hand side of the truncated asymptotic ansatz $u^{\varepsilon,n}$; see (12)–(15). The following lemma is in effect an implication of the above described formal asymptotic construction: it is supplemented by a more accurate bookkeeping of the structure of the remainder term $R^{\varepsilon,n}$ (as needed for purposes of this work), which is bound, by the construction, to contain only the terms of orders ε^{n+1} and ε^{n+2} for fixed n and small ε; cf. [3, 10].

LEMMA 3. *Under the assumptions of Theorem* 1 *one has* $L^\varepsilon u^{\varepsilon,n} = f + R^{\varepsilon,n}$ *with* $R^{\varepsilon,n} \in H^{-1}(\mathbb{T})$, *and*

$$\begin{aligned}R^{\varepsilon,n} = -\varepsilon^{n+1} \Bigg(&\sum_{l=0}^{n+2} \sum_{|k|=l} (\,(A_{ij}N_k)_{,j} + A_{ij}N_{k,j}) D_{x_i} D_x^k v_{n-l+2} \\ &+ \sum_{l=0}^{n+1} \sum_{|k|=l} A_{ij} N_k D_{x_i x_j} D_x^k (v_{n-l+1} + \varepsilon v_{n-l+2}) \\ &+ \varepsilon \sum_{|k|=n+2} A_{ij} N_k D_{x_i x_j} D_x^k v_0 \Bigg)\end{aligned} \tag{45}$$

(denoting $D_{x_i} := \partial/\partial x_i$, $D_{x_i x_j} := \partial^2/(\partial x_i \partial x_j)$*).*

Proof. The proof is a straightforward calculation by substituting the expansion (14), (15) into (4). We notice that since $A_{ij} \in L^\infty(\mathbb{T})$, $N_k \in H^1(\mathbb{T})$, and $v_s \in C^\infty(\mathbb{T})$, all the "product" terms in (45) are in $H^{-1}(\mathbb{T})$. For example,

$$\begin{aligned} ((A_{ij}N_k)_{,j})\left(\frac{x}{\varepsilon}\right) D_{x_i} D_x^k V^{(n-l+2)}(x,\varepsilon) &= \varepsilon \frac{\partial}{\partial x_j}\Big((A_{ij}N_k)\left(\frac{x}{\varepsilon}\right) D_{x_i} D_x^k V^{(n-l+2)}(x,\varepsilon)\Big) \\ &\quad -\varepsilon\Big((A_{ij}N_k)\left(\frac{x}{\varepsilon}\right) D_{x_i x_j} D_x^k V^{(n-l+2)}(x,\varepsilon)\Big), \end{aligned} \tag{46}$$

with the first term in the latter expression being a derivative of an L^2 function and the last one an L^2 function itself.

The terms up to order $O(\varepsilon^n)$ equal f by (14), (15), (30), (32), and (33). Via direct inspection,

$$\begin{aligned} (L^\varepsilon u^{\varepsilon,n})(x) &= -\nabla\cdot\Big(A\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon,n}\Big)(x) \\ &= -\nabla\cdot\left(A\left(\frac{x}{\varepsilon}\right)\nabla\sum_{l=0}^{n+2}\varepsilon^l\sum_{|k|=l} N_k\left(\frac{x}{\varepsilon}\right) D_x^k\, V^{(n-l+2)}(x,\varepsilon)\right) \\ &= \sum_{l=0}^{n+2}\varepsilon^{l-2}\sum_{|k|=l}(L_y^1 N_k)\left(\frac{x}{\varepsilon}\right) D_x^k V^{(n-l+2)}(x,\varepsilon) \\ &\quad -\sum_{l=0}^{n+2}\varepsilon^{l-1}\sum_{|k|=l}((A_{ij}N_k)_{,j}+A_{ij}N_{k,j})\left(\frac{x}{\varepsilon}\right) D_{x_i} D_x^k V^{(n-l+2)}(x,\varepsilon) \\ &\quad -\sum_{l=0}^{n+2}\varepsilon^{l}\sum_{|k|=l}(A_{ij}N_k)\left(\frac{x}{\varepsilon}\right) D_{x_i x_j} D_x^k V^{(n-l+2)}(x,\varepsilon). \end{aligned}$$

Now we replace $V^{(n-l+2)}(x,\varepsilon)$ by $V^{(n-l+1)}(x,\varepsilon)+\varepsilon^{n-l+2}v_{n-l+2}$ in the second term and by $V^{(n-l)}(x,\varepsilon)+\varepsilon^{n-l+1}v_{n-l+1}+\varepsilon^{n-l+2}v_{n-l+2}$ in the last term; see (15). These "remainders" containing v_{n-l+1} and v_{n-l+2} as well as the term corresponding to $l=n+2$ in the last sum, all being of order ε^{n+1} and ε^{n+2}, produce exactly $R^{\varepsilon,n}$. Therefore we have

$$\begin{aligned} (L^\varepsilon u^{\varepsilon,n})(x) &= R^{\varepsilon,n}+\sum_{l=0}^{n+2}\varepsilon^{l-2}\sum_{|k|=l}(L_y^1 N_k)\left(\frac{x}{\varepsilon}\right) D_x^k V^{(n-l+2)}(x,\varepsilon) \\ &\quad -\sum_{l=0}^{n+1}\varepsilon^{l-1}\sum_{|k|=l}((A_{ij}N_k)_{,j}+A_{ij}N_{k,j})\left(\frac{x}{\varepsilon}\right) D_{x_i} D_x^k V^{(n-l+1)}(x,\varepsilon) \\ &\quad -\sum_{l=0}^{n}\varepsilon^{l}\sum_{|k|=l}(A_{ij}N_k)\left(\frac{x}{\varepsilon}\right) D_{x_i x_j} D_x^k V^{(n-l)}(x,\varepsilon). \end{aligned}$$

Now we change the summation indices in the two latter terms to obtain the first $n+2$ terms of the series (21)

$$\begin{aligned} (L^\varepsilon u^{\varepsilon,n})(x) &= R^{\varepsilon,n}+\sum_{l=2}^{n+2}(L_y^1 N_k - T_k) D^k V^{(n-l+2)} \\ &= R^{\varepsilon,n}-\sum_{l=2}^{n+2}\varepsilon^{l-2}\sum_{|k|=l} h_k D^k V^{(n-l+2)} = f + R^{\varepsilon,n}, \end{aligned}$$

having used in the last equality (30)–(33). ∎

Using the above formula (45) for the remainder term, we estimate $R^{\varepsilon,n}$ with an explicit dependence on both ε and n as in the following lemma ($[\cdot]$ denotes the entire part).

LEMMA 4. *Under the assumptions of Theorem* 1, *there exist* C_3, ε_0 *such that for all* n *and all* $0 < \varepsilon < \varepsilon_0$ *the remainder term* $R^{\varepsilon,n}$ *can be estimated as follows:*

$$\|R^{\varepsilon,n};\ H^{-1}(\mathbb{T})\| \le C_3^n \varepsilon^{n+1} \|f;\ H^{n+5+[d/2]}(\mathbb{T})\|. \tag{47}$$

Proof. Here we combine the formula for the error term in Lemma 3 with the estimates in Lemma 2. To estimate the H^{-1} norm of the aggregates like $((A_{ij}N_k)D^{|k|+1}v)_{,j}$ (see (46)) and $A_{ij}N_kD^{|k|+2}v$ we need to ensure that $D^{|k|+2}v$ is in L^∞. By the Sobolev embedding theorems this holds if $v \in H^{|k|+3+[d/2]}$. Therefore

$$\begin{aligned}
\|R^{\varepsilon,n};\ H^{-1}(\mathbb{T})\| = \Bigg\| \varepsilon^{n+1} \Bigg(& \sum_{l=0}^{n+2} \sum_{|k|=l} \big((A_{ij}N_k)_{,j} + A_{ij}N_{k,j}\big) D_{x_i} D_x^k v_{n-l+2} \\
& + \sum_{l=0}^{n+1} \sum_{|k|=l} A_{ij}N_k D_{x_i,x_j} D_x^k (v_{n-l+1} + \varepsilon v_{n-l+2}) \\
& + \varepsilon \sum_{|k|=n+2} A_{ij}N_k D_{x_i,x_j} D_x^k v_0 \Bigg);\ H^{-1}(\mathbb{T}) \Bigg\| \\
\le \varepsilon^{n+1} \Bigg(& C \sum_{l=0}^{n+2} l^{d-1} \|A; L^\infty\| \mathcal{N}^{(l)} \mathcal{V}^{(n-l+2,l+3+[d/2])} \\
& + C \sum_{l=0}^{n+1} l^{d-1} \|A; L^\infty\| \mathcal{N}^{(l)} \Big(\mathcal{V}^{(n-l+1,l+3+[d/2])} \\
& \qquad + \varepsilon \mathcal{V}^{(n-l+2,l+3+[d/2])} \Big) \\
& + C\varepsilon n^{d-1} \|A; L^\infty\| \mathcal{N}^{(n+2)} \mathcal{V}^{(0,n+5+[d/2])} \Bigg) \\
\le C\varepsilon^{n+1} & \Big(\max\{M_{\mathcal{N}}, M_{\mathcal{V}}\} \Big)^{n+2} n^d \|f; H^{n+5+[d/2]}\|.
\end{aligned}$$

Here we have used again the Poincaré inequality (42). An appropriate choice of C_3 yields the result. □

Remark 1. The last lemma could also be used to rederive results for finite regularity f, or smooth f, which are not necessarily in any Gevrey space $\mathcal{G}^\beta$. If, for example, f has finite regularity, i.e., $f \in H^M(\mathbb{T})$ and $f \notin H^{M+1}(\mathbb{T})$ for some M, then, as the above procedure demonstrates, only a finite number of terms in the asymptotic expansion can be constructed, and the H^{-1} norms can be bounded only for $n < M - 4 - d/2$. If, however, $f \in C^\infty(\mathbb{T})$, but no assumptions are made on the rate of growth of its H^l norms for large l, the estimate (47) still holds for any n, but there is no control over the growth of the Sobolev norms of f with n in the right-hand side of (47). The latter would prevent us from improving the polynomial "asymptotics in all orders" any further. This highlights the importance of the Gevrey extreme regularity of f for the exponential error bounds.

5. Proof of Theorem 1. The proof of the theorem is now essentially a corollary of Lemma 4, the estimates (6) holding due to the assumption of Gevrey regularity of f and standard elliptic regularity theory. Let us first introduce a "normalized" approximation

$$\tilde{u}^{\varepsilon,n} := u^{\varepsilon,n} - \langle u^{\varepsilon,n}\rangle. \tag{48}$$

By the elliptic regularity theory for all n we have

$$\|\tilde{u}^{\varepsilon} - u^{\varepsilon,n}\ ;\ H^1(\mathbb{T})\| \le C\|R^{\varepsilon,n};\ H^{-1}(\mathbb{T})\|. \tag{49}$$

Using Lemma 4, we obtain

$$\|\tilde{u}^{\varepsilon} - u^{\varepsilon,n}\ ;\ H^1(\mathbb{T})\| \le CC_3^n\varepsilon^{n+1}\|f; H^{n+4+[d/2]}\|. \tag{50}$$

Let us next show that the mean $\langle u^{\varepsilon,n}\rangle$ can also be estimated in a similar way. Due to representations (12), (13) we have

$$\langle u^{\varepsilon,n}\rangle = \sum_{m=0}^{n+2}\varepsilon^m\sum_{l=0}^{m}\sum_{|k|=l}\left\langle N_k\left(\frac{x}{\varepsilon}\right)D_x^k v_{m-l}(x)\right\rangle. \tag{51}$$

Note that $\langle N_k\rangle = 0$; therefore for any $s > 0$ the functions $\left((-\Delta_y)^{-s}N_k(\cdot)\right)(y)$ and $\left((-\Delta_x)^{-s}N_k(\frac{\cdot}{\varepsilon})\right)(x)$ are correctly defined functions with zero mean, using, for example, the Fourier representation for $(-\Delta_y)$ on a torus $\mathbb{T}$. Moreover, they are linked via

$$\left((-\Delta_x)^{-s}N_k\left(\frac{\cdot}{\varepsilon}\right)\right)(x) = \varepsilon^{2s}\left((-\Delta_y)^{-s}N_k(\cdot)\right)\left(\frac{x}{\varepsilon}\right) \tag{52}$$

(recall $\varepsilon^{-1}\in\mathbb{N}$). Thus, integrating (51) by parts sufficiently many times, we get

$$\begin{aligned}(53)\ &\langle u^{\varepsilon,n}\rangle \\ &= \sum_{m=0}^{n+2}\varepsilon^m\sum_{l=0}^{m}\sum_{|k|=l}\varepsilon^{n+2-m}\left\langle\left((-\Delta_y)^{-\frac{n+2-m}{2}}N_k\right)\left(\frac{x}{\varepsilon}\right)(-\Delta_x)^{\frac{n+2-m}{2}}D_x^k v_{m-l}(x)\right\rangle.\end{aligned}$$

Now, via the Cauchy–Schwartz inequality,

(54)

$$|\langle u^{\varepsilon,n}\rangle| \le \varepsilon^{n+2}\sum_{m=0}^{n+2}\sum_{l=0}^{m}\sum_{|k|=l}\|(-\Delta_y)^{-\frac{n+2-m}{2}}N_k;\ L^2(\mathbb{T})\|\,\|v_{m-l};\ H^{n+2-m+l}(\mathbb{T})\|.$$

Therefore, applying the Poincaré inequality (42) to the first norm and then using estimates (36), (37) of Lemma 2, we have

$$\begin{aligned}|\langle u^{\varepsilon,n}\rangle| &\le \varepsilon^{n+2}\sum_{m=0}^{n+2}\sum_{l=0}^{m}Cl^{d-1}\left(\sup_{|k|=l}\|N_k;\ H^1\|\right)\sum_{p=0}^{m-l}M_{\mathcal{V}}^{p+1}\|f; H^{n+2-m+l+p}\| \\ &\le \varepsilon^{n+2}\|f; H^{n+2}\|\sum_{m=0}^{n+2}\sum_{l=0}^{m}Cl^dM_{\mathcal{N}}^l\sum_{p=0}^{m-l}M_{\mathcal{V}}^{p+1} \\ &\le \varepsilon^{n+2}\|f; H^{n+2}\|\sum_{m=0}^{n+2}\sum_{l=0}^{m}Cl^dM_{\mathcal{N}}^lM_{\mathcal{V}}^{m-l},\end{aligned}$$

and therefore

$$|\langle u^{\varepsilon,n}\rangle| \le \varepsilon^{n+2} C C_4^n \|f; H^{n+2}\|. \tag{55}$$

Combining the latter with (50) we finally get

$$\|u^\varepsilon - u^{\varepsilon,n} \ ; \ H^1(\mathbb{T})\| \le C C_5^n \varepsilon^{n+1} \|f; H^{n+5+[d/2]}\| \tag{56}$$

for small enough ε with an appropriate constant $C_5 > 0$.

Further, by (6), $\|f; H^{n+5+[d/2]}\| \le B^{n+5+[d/2]}((n+5+[d/2])!)^\beta$. Using the Stirling formula (11) for the factorial (implying $M! = \Gamma(M+1) \le C M^{M+1/2} e^{-M}$ for any $M \in \mathbb{N}$ with some $C > 0$), we obtain

$$\begin{aligned} &\|u^\varepsilon - u^{\varepsilon,n} \ ; \ H^1(\mathbb{T})\| \\ &\le C C_5^n \varepsilon^{n+1} B^{n+5+[d/2]} (n+5+[d/2])^{(n+5+[d/2])\beta} e^{-\beta(n+5+[d/2])} n^{1/2} \\ &\le C\varepsilon^{n+1} (C_6 B)^n n^{n\beta} = C\varepsilon \exp(n \ln(n^\beta C_6 B \varepsilon)). \end{aligned}$$

Thus, we get the desired decay of this norm if the logarithm in the latter exponent is uniformly negative. The latter can be assured by choosing $n(\varepsilon) \in (\kappa_1 \varepsilon^{-1/\beta}, \kappa_2 \varepsilon^{-1/\beta})$ with any choice of constants κ_1 and κ_2 such that $0 < \kappa_1 < \kappa_2 < (C_6 B)^{-1/\beta}$. Indeed, we then estimate

$$\|u^{\varepsilon,n(\varepsilon)} - u^\varepsilon \ ; \ H^1(\mathbb{T})\| \le C\varepsilon \exp[(\kappa_1 \ln(\kappa_2^\beta C_6 B))\varepsilon^{-1/\beta}],$$

which implies (17) by choosing $C_1 = C$ and $C_2 = -\kappa_1 \ln(\kappa_2^\beta C_6 B) > 0$. The theorem is proved. ∎

Remark 2 (on the proof of Theorem 1′). The proof of Theorem 1′ conceptually follows the above proof of Theorem 1. We briefly sketch the proof emphasizing only the most significant alterations to the above argument. First note that, although we still use the asymptotic series (12) for the approximation, its precise structure slightly differs from (14); namely, (12) is now represented in the following form:

$$u^{\varepsilon,n}(x) = \sum_{l=0}^{n+2} \varepsilon^l \sum_{s=0}^{[\frac{l}{2}]} \sum_{|k|=l-2s} N_k^{(s)}\left(\frac{x}{\varepsilon}\right) D_x^k V^{(n-l+2)}(x,\varepsilon). \tag{57}$$

For $N_k^{(s)}$, analogously to (25), (26), one deduces the recurrence relations

$$L_y^1 N_k^{(s)} = T_k^{(s)}(y) - h_k^{(s)} - N_k^{(s-1)}, \qquad \langle N_k^{(s)}\rangle = 0, \ s \ge 0, \tag{58}$$

assuming henceforth that $N_k^{(-1)} \equiv 0$. If $|k| \ge 2$, then one finds

$$T_k^{(s)} = \sum_{\substack{i,j=1,\dots,d \\ k'=k-e_i\ge 0}} \left((A_{ij} N_{k'}^{(s)})_{,j} + A_{ij} N_{k',j}^{(s)}\right) + \sum_{\substack{i,j=1,\dots,d \\ k''=k-e_i-e_j\ge 0}} A_{ij} N_{k''}^{(s)}, \tag{59}$$

and otherwise

$$|k| = 1, k = e_i: \quad T_k^{(s)} = (A_{ij} N_0^{(s)})_{,j} + A_{ij} N_{0,j}^{(s)}, \tag{60}$$

$$k = 0: \qquad T_k^{(s)} = 0, \ \ N_0^{(0)} \equiv 1. \tag{61}$$

Further, in all the cases $h_k^{(s)} = \langle T_k^{(s)} \rangle$, except $h_0^{(1)} = -1$. Obviously one has $N_k^{(0)} = N_k$ (see (22)–(26)), and thus by induction in s one finds all $N_k^{(s)}$. Now let us introduce $\mathfrak{N}_q = \max_{|k|+2s=q} |N_k^{(s)}|_1$. Due to (59) we obviously have $|T_k^{(s)} - h_k^{(s)}|_{-1} \leq C(\mathfrak{N}_{|k|+2s-1} + \mathfrak{N}_{|k|+2s-2})$. The basic elliptic estimate (38) for the problem (58) still holds and therefore implies that $\mathfrak{N}_q \leq M_1 \mathfrak{N}_{q-1} + M_2 \mathfrak{N}_{q-2}$, which gives an exponential estimate of growth of $\mathfrak{N}_q$: with large enough $M_{\mathfrak{N}}$ for all $q \geq 0$

$$\mathfrak{N}_q \leq (M_{\mathfrak{N}})^q. \tag{62}$$

Turning now to evaluation of $v_s(x)$, substituting the expansion (57) into (18) we observe that $V^{(\infty)}(x, \varepsilon)$ formally satisfies

$$\left(-A_{ij}^{\text{hom}} D_{ij} + 1 + \sum_{l=3}^{\infty} \varepsilon^{l-2} \sum_{s=0}^{[\frac{l}{2}]} \sum_{|k|=l-2s} h_k^{(s)} D_x^k \right) V^{\infty}(x, \varepsilon) = f(x) \quad \text{in } \mathbb{R}^d; \tag{63}$$

therefore we have

$$-A_{ij}^{\text{hom}} v_{s,ij} + v_s = f_s \quad \text{in } \mathbb{R}^d, \tag{64}$$

where

$$f_0 = f, \tag{65}$$

$$f_s = \sum_{l=3}^{s+2} \sum_{r=0}^{[\frac{l}{2}]} \sum_{|k|=l-2r} h_k^{(r)} D_x^k v_{s-l+2}, \quad s \geq 1. \tag{66}$$

The latter differs from (33) only by the presence of lower order derivatives, and without any significant alteration one deduces an exponential estimate (37) in very much the same way as in Lemma 2. As a result, introducing $\mathfrak{V}^{(s,m)} = \|v_s;\ H^m(\mathbb{R}^d)\|$ with large enough $M_{\mathfrak{V}}$, we get by induction in s an estimate

$$\mathfrak{V}^{(s,m)} \leq \sum_{k=0}^{s} (M_{\mathfrak{V}})^{k+1} \|f\ ;\ H^{m+k}(\mathbb{R}^d)\|. \tag{67}$$

The remainder estimate is bound to be of order ε^{n+1}, and with some minor technical alteration of the argument in section 4 one also gets

$$\|R^{\varepsilon,n}; H^{-1}(\mathbb{R}^d)\| \leq C\varepsilon^{n+1} M^n \|f; H^{n+5+[d/2]}(\mathbb{R}^d)\|. \tag{68}$$

Finally, repeating the argument at the beginning of this section (omitting the consideration of the mean), employing appropriate modifications of the Poincaré inequality, ellipticity estimates, Sobolev embedding, etc. from (68), and the fact that $f \in \mathcal{G}^{\beta}(\mathbb{R}^d)$, one finally deduces Theorem $1'$.

Remark 3. Note that as formulated the theorems admit some further sharpening: for example, one can replace $H^1(\mathbb{T})$ norm in (17) with $W^{1,p}(\mathbb{T})$ norm, where $p \in (2, p_0(d, \nu_0))$ with some $p_0(d, \nu_0) > 2$. Indeed, as can be seen from the structure of our argument, we select a functional space according to the fundamental ellipticity estimate (38), whereas the latter (38) can be refined in the case of bounded measurable coefficients and the right-hand side being the divergence of an L^{∞} (vector-)function (see, e.g., [22, Chapter 6]).

Remark 4. The proof of the theorems has been via a straightforward "bookkeeping" of the terms in the full two-scale asymptotic expansion (20)–(29). On the other hand, this is known to be related to the so-called spectral method in homogenization and closely related "Bloch approximation" approach; see, e.g., [24, 25, 26, 27, 28, 6]. There is no doubt that these spectral methods are capable of at least reestablishing the background results on the "homogenization in all orders" (i.e., approximations with arbitrary high order polynomial error bounds). An interesting further prospect would be to interpret the results presented here on *exponential* homogenization in terms of underlying analytic spectral properties of the Floquet–Bloch operator with periodic coefficients.

6. On the optimality of the exponential error (17): An example. In this section we demonstrate that the main exponential error bound (17) of Theorem 1 for $\|u^\varepsilon - u^{\varepsilon,n}\ ;\ H^1(\mathbb{T})\|$ is "optimal" for a particular class of one-dimensional examples. Namely, we show that by whatever choice of the truncation $n(\varepsilon)$ the error bound (17) cannot be improved apart from "optimizing" the choice of constants C_1 and C_2. This is done by proving an analogous exponential lower bound for the error; see (79). The latter is obtained by an optimal truncation $n(\varepsilon)$ of lower bounds derived for each n and ε, which in turn is observed to be delivered by $n(\varepsilon)$ within the range (16). In this sense the exponential error bound (17) is sharp.

We consider the following one-dimensional example. Consider elliptic problem

$$-\frac{d}{dx}\left(a(x/\varepsilon)\frac{d}{dx}u(x)\right) = f(x), \tag{69}$$

with one-periodic boundary conditions, $\langle u\rangle = \langle f\rangle = 0$, $\varepsilon = 1/N$, $N \in \mathbb{N}$, which is the one-dimensional version of the problem (4)–(5), with unique solution $u^\varepsilon(x)$. To be specific, let us consider[2]

$$a(y) = \frac{1}{3/2 - \cos(2\pi y - \pi/4)}. \tag{70}$$

We fix arbitrary $\beta \geq 1$ and assume the right-hand side f to be an infinitely differentiable real-valued 1-periodic function with real nonnegative Fourier coefficients f_k (hence $f_{-k} = f_k$), i.e.,

$$f(x) = \sum_{k\in\mathbb{Z},k\neq 0} f_k \exp(2i\pi kx) = 2\sum_{k=1}^{\infty} f_k \cos(2\pi kx), \quad f_k \geq 0. \tag{71}$$

We further assume that f satisfies the "converse" inequality to (6) determining β-Gevrey regular functions; i.e., there exists $b > 0$ such that

$$\|f\ ;\ H^l(\mathbb{T})\| \geq b^l (l!)^\beta \quad \text{for all } l \in \mathbb{N}. \tag{72}$$

In particular, for "sharp" β-Gevrey regular functions both (6) and (72) hold simultaneously:

$$b^l(l!)^\beta \leq \|f\ ;\ H^l(\mathbb{T})\| \leq B^l(l!)^\beta, \quad 0 < b \leq B < +\infty, \quad \text{for all } l \in \mathbb{N}. \tag{73}$$

[2]The analysis of this section can be generalized in a straightforward way to more general $a(y)$, for example, $a(y) = (a_0 - a_1\cos(2\pi y) - a_2\sin(2\pi y))^{-1}$, $a_0 > 0$, $a_1 \neq 0$, $a_2 \neq 0$, $a_1^2 + a_2^2 < a_0^2$. We do not pursue maximal generality to avoid unnecessary further algebraic complications.

A sufficient condition for f to satisfy (73) is for its Fourier coefficients f_k to decay exponentially with the "rate" $|k|^{1/\beta}$, i.e.,

$$A_1 \exp(-B_1|k|^{1/\beta}) \leq f_k \leq A_2 \exp(-B_2|k|^{1/\beta}) \tag{74}$$

with positive A_1, A_2, B_1, and B_2.

To see that such f satisfies (73) one can first apply the Plancherel theorem to definition (7), implying

$$\begin{aligned} (2\pi)^{2l} A_1 \sum_{k\in\mathbb{N}} \exp(-2B_1 k^{1/\beta}) k^{2l} &\leq (2\pi)^{2l} \sum_{k\in\mathbb{N}} k^{2l} f_k^2 \leq \|f;\ H^l(\mathbb{T})\|^2 \\ &\leq 4(2\pi)^{2l} \sum_{k\in\mathbb{N}} k^{2l} f_k^2 \leq 4(2\pi)^{2l} A_2 \sum_{k\in\mathbb{N}} \exp(-2B_2 k^{1/\beta}) k^{2l}. \end{aligned} \tag{75}$$

Then one can notice that the sums in (75) can be further bounded both from above and from below as follows: there exist l-independent positive constants D_1 and D_2 such that

$$\begin{aligned} D_1\beta(2B_1)^{-\beta(2l+1)}\Gamma((2l+1)\beta) = D_1 \int_0^\infty \exp(-2B_1 s^{1/\beta}) s^{2l} ds \\ \leq \sum_{k\in\mathbb{N}} \exp(-2B_1 k^{1/\beta}) k^{2l} \leq \sum_{k\in\mathbb{N}} \exp(-2B_2 k^{1/\beta}) k^{2l} \\ \leq D_2 l \int_0^\infty \exp(-2B_2 s^{1/\beta}) s^{2l} ds = D_2 l\beta(2B_2)^{-\beta(2l+1)}\Gamma((2l+1)\beta). \end{aligned} \tag{76}$$

(A way to establish (76) is by noticing that the series, asymptotically for large l, coincides to the main order with the integral.) Finally, by the application of the Stirling formula (11) we obtain $D_3^{2l}(l!)^{2\beta} \leq \Gamma((2l+1)\beta) \leq D_4^{2l}(l!)^{2\beta}$ with l-independent D_3, D_4, which implies (73).

An example of a function f satisfying (71) and (73) is

$$f(x) = \sum_{k\in\mathbb{Z}, k\neq 0} \exp(-|k|^{1/\beta}) \exp(2i\pi kx). \tag{77}$$

In particular, for $\beta = 1$

$$f(x) = 2\,\mathrm{Re} \sum_{k=1}^{\infty} \exp(k(2i\pi x - 1)) = \frac{2e\cos(2\pi x) - 2}{(e^2+1) - 2e\cos(2\pi x)} \tag{78}$$

is clearly analytic, with poles at $x = \pm i/(2\pi) + n$, $n \in \mathbb{Z}$.

We formulate the following optimality theorem for the above one-dimensional case.

THEOREM 5. *For any f satisfying* (74) *with $B_1 = B_2$ there exist positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that the following lower error bound for the exact solution u^ε of the problem* (69)–(70) *and its asymptotic approximation $u^{\varepsilon,n}$ holds for any $n \in \mathbb{N}$ and any $\varepsilon = 1/N$, $N \in \mathbb{N}$:*

$$\|u^\varepsilon - u^{\varepsilon,n}\ ;\ H^1(\mathbb{T})\| \geq \tilde{C}_1 \exp(-\tilde{C}_2 \varepsilon^{-\frac{1}{\beta}}). \tag{79}$$

Proof. In the one-dimensional case, the general recurrence relations (22)–(27), (30)–(33) for the correctors N_k, the "homogenized coefficients" h_k, and "slowly varying" parts v_s specialize to simple ODEs (see, e.g., [23, section 1F]), which can be solved explicitly.[3] In particular, the equations (25)–(26) for the "main corrector" N_1 specialize to

$$\frac{d}{dy}N_1(y) = \frac{a^{-1}(y)}{\langle a^{-1}\rangle} - 1 = \frac{2}{3}\left(\frac{3}{2} - \cos\left(2\pi y - \frac{\pi}{4}\right)\right) - 1 = -\frac{2}{3}\cos\left(2\pi y - \frac{\pi}{4}\right), \tag{80}$$

implying $N_1(y) = -(3\pi)^{-1}\sin(2\pi y - \pi/4)$. All the higher order correctors N_k, $k \geq 2$, have a similar form, due to the recurrence relations

$$\frac{d}{dy}N_k = -N_{k-1} \tag{81}$$

(the latter also immediately follows by direct substitution of (20) into (69)). As a result,

$$\begin{aligned} N_{2m} &= (-1)^m \frac{\cos(2\pi y - \pi/4)}{3\pi(2\pi)^{2m-1}} = (-1)^m 2^{1/2}\frac{\cos(2\pi y) + \sin(2\pi y)}{3(2\pi)^{2m}}, \quad m \geq 1, \\ N_{2m+1} &= (-1)^{m+1}\frac{\sin(2\pi y - \pi/4)}{3\pi(2\pi)^{2m}} = (-1)^m 2^{1/2}\frac{\cos(2\pi y) - \sin(2\pi y)}{3(2\pi)^{2m+1}}, \quad m \geq 0. \end{aligned} \tag{82}$$

Further, v_0 is given by homogenized equation (30) ($s = 0$) specializing in the one-dimensional case to

$$-h_2\frac{d^2}{dx^2}v_0 = f, \tag{83}$$

where $h_2 = A^{\text{hom}} = \langle a^{-1}\rangle^{-1} = 2/3$. Furthermore, in the one-dimensional case $h_k = 0$ for all $k \geq 3$ via a straightforward analysis of the recurrent relations (23)–(27) (see, e.g., [23, section 1F]). The latter immediately implies via (30) and (33) that $v_k = 0$ for all $k \geq 1$. Taking the above into account specializes the remainder term (45) in Lemma 3 to

$$\begin{aligned} R^{\varepsilon,n}(x) &= -\varepsilon^{n+1}\left(\left(aN'_{n+2} + a'N_{n+2}\right)\left(\frac{x}{\varepsilon}\right)D^{n+3}v_0(x)\right. \\ &\qquad\left. +\varepsilon a\left(\frac{x}{\varepsilon}\right)N_{n+2}\left(\frac{x}{\varepsilon}\right)D^{n+4}v_0(x)\right) \\ &= -\frac{d}{dx}\left(\varepsilon^{n+2}a\left(\frac{x}{\varepsilon}\right)N_{n+2}\left(\frac{x}{\varepsilon}\right)D^{n+3}v_0(x)\right) = -\frac{d}{dx}\Phi^{\varepsilon,n}(x), \end{aligned} \tag{84}$$

where

$$\Phi^{\varepsilon,n}(x) := \varepsilon^{n+2}a\left(\frac{x}{\varepsilon}\right)N_{n+2}\left(\frac{x}{\varepsilon}\right)D^{n+3}v_0(x). \tag{85}$$

Employing in the above the explicit solutions (82) for N_k and (83) for v_0, we arrive at

$$\begin{aligned} \Phi^{\varepsilon,n}(x) &= \frac{(-1)^{n/2}\varepsilon^{n+2}(\cos(2\pi x/\varepsilon) + \sin(2\pi x/\varepsilon))}{2^{1/2}(2\pi)^{n+2}\left[3/2 - \cos(2\pi x/\varepsilon - \pi/4)\right]}D^{n+1}f(x), \quad n = 2m,\ m \geq 1, \quad (86) \\ \Phi^{\varepsilon,n}(x) &= \frac{(-1)^{(n+1)/2}\varepsilon^{n+2}(\cos(2\pi x/\varepsilon) - \sin(2\pi x/\varepsilon))}{2^{1/2}(2\pi)^{n+2}\left[3/2 - \cos(2\pi x/\varepsilon - \pi/4)\right]}D^{n+1}f(x), \quad n = 2m+1,\ m \geq 0. \quad (87) \end{aligned}$$

[3] We remark that the present one-dimensional case is integrable, and an alternative but related approach for analyzing the error term in the asymptotics is from the exact solution; see Remark 5.

From (84), by the definition of the H^{-1} norm, we have

$$\|R^{\varepsilon,n};\ H^{-1}\|^2 \geq C\|\Phi^{\varepsilon,n} - \langle\Phi^{\varepsilon,n}\rangle\,;\ L^2\|^2 = C\left(\|\Phi^{\varepsilon,n};\ L^2\|^2 - \left|\int_0^1 \Phi^{\varepsilon,n}(x)dx\right|^2\right) \tag{88}$$

(recalling that C denotes constants whose precise value is insignificant).

With the aim of further bounding (88) from below, we introduce for any given $\varepsilon = 1/N$ and n functions $h_N(x)$ as follows:

$$\begin{aligned} h_N(x) = h(Nx) &= \frac{\cos(2\pi Nx) + \sin(2\pi Nx)}{[3/2 - \cos(2\pi Nx - \pi/4)]},\ N = 2m, \\ h_N(x) = h(Nx) &= \frac{\cos(2\pi Nx) - \sin(2\pi Nx)}{[3/2 - \cos(2\pi Nx - \pi/4)]},\ N = 2m+1. \end{aligned} \tag{89}$$

We prove the following lemma.

LEMMA 6. *There exists a constant C such that for any f satisfying* (71) *and for all n and N*

$$\|h_N D^{n+1} f;\ L^2\|^2 \geq C\|D^{n+1} f;\ L^2\|^2. \tag{90}$$

Proof. Choosing first n to be even, $n = 2m$, notice that

$$\begin{aligned} \|h_N(x)D^{n+1}f(x);\ L^2\|^2 &:= \int_0^1 \frac{(\cos(2\pi Nx) + \sin(2\pi Nx))^2}{[3/2 - \cos(2\pi Nx - \pi/4)]^2}\left(D^{n+1}f\right)^2 dx \\ &\geq \frac{4}{25}\int_0^1 (\cos(2\pi Nx) + \sin(2\pi Nx))^2 \left(D^{n+1}f\right)^2 dx \\ &= \frac{4}{25}\|\left(\cos(2\pi Nx) + \sin(2\pi Nx)\right) D^{n+1}f(x);\ L^2\|^2. \end{aligned}$$

We next notice that for n even and for f given by (71) $D^{n+1}f$ is represented by a sine Fourier series, implying that $\cos(2\pi Nx)D^{n+1}f(x)$ and $\sin(2\pi Nx)D^{n+1}f(x)$ are orthogonal in $L^2(0,1)$ and hence

$$\|\left(\cos(2\pi Nx) + \sin(2\pi Nx)\right) D^{n+1}f(x);\ L^2\|^2 \geq \|\cos(2\pi Nx)D^{n+1}f(x);\ L^2\|^2.$$

Further,

$$\begin{aligned} g_{Nn}(x) := \cos(2\pi Nx)D^{n+1}f(x) &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left(e^{2i\pi Nx} + e^{-2i\pi Nx}\right)(2i\pi k)^{n+1} f_k e^{2i\pi kx} \\ &= \frac{1}{2}\sum_{m\in\mathbb{Z}} e^{2i\pi mx}\left[(i2\pi(m-N))^{n+1} f_{m-N} + (i2\pi(m+N))^{n+1} f_{m+N}\right]. \end{aligned}$$

Hence, applying the Plancherel theorem,

$$\begin{aligned}\|g_{Nn}(x);\ L^2\|^2 &= \frac{(2\pi)^{2n+2}}{4}\sum_{m\in\mathbb{Z}}\left[(m-N)^{n+1}f_{m-N}+(m+N)^{n+1}f_{m+N}\right]^2\\ &\geq \frac{(2\pi)^{2n+2}}{4}\sum_{m=N+1}^{\infty}\left[(m-N)^{n+1}f_{m-N}+(m+N)^{n+1}f_{m+N}\right]^2\\ &\geq \frac{(2\pi)^{2n+2}}{4}\sum_{m=N+1}^{\infty}\left((m-N)^{n+1}f_{m-N}\right)^2 = \frac{(2\pi)^{2n+2}}{4}\sum_{m=1}^{\infty}\left(m^{n+1}f_m\right)^2\\ &= \frac{(2\pi)^{2n+2}}{8}\sum_{m\in\mathbb{Z}}\left(m^{n+1}f_m\right)^2 = \frac{1}{8}\|D^{n+1}f(x);\ L^2\|^2.\end{aligned}$$

In the latter we have used the nonnegativity of Fourier coefficients f_k, their symmetry ($f_{-k}=f_k$), and the fact that $\langle f\rangle = 0$ (hence $f_0 = 0$).

The above proves the lemma for even n. The proof for odd n is fully analogous, with the sign alteration between the "sine" and "cosine" terms, then noticing that $D^{n+1}f$ is represented by a cosine Fourier series, and then using the orthogonality and neglecting the sine term as before. ∎

We next aim at showing that, at least for sufficiently large n, the last term in the right-hand side of (88) can be bounded from the above as in (90) but with a smaller constant.

LEMMA 7. *For any f satisfying* (74) *with $B_1 = B_2$ there exists $n_0 > 0$ such that for all $n > n_0$ and all N*

$$\langle h_N(x)D^{n+1}f(x)\rangle^2 := \left|\int_0^1 h_N(x)D^{n+1}f(x)dx\right|^2 \leq \frac{1}{2}C\|D^{n+1}f(x);\ L^2\|^2, \tag{91}$$

where C is same as in Lemma 6.

Proof. The Fourier series of $h_N(x)$ has the form

$$h_N(x) = \sum_{\ell\in\mathbb{Z}} h_\ell \exp(i2\pi N\ell x), \tag{92}$$

with $\{h_\ell\}$ being two possible sets of (rapidly decaying) Fourier coefficients (for N even and odd, according to (89)), independent of N for all $\ell\in\mathbb{Z}$. Then,

$$\begin{aligned}\langle h_N(x)D^{n+1}f(x)\rangle^2 &= \left|\sum_{\ell\in\mathbb{Z}} h_\ell(2\pi N\ell)^{n+1}f_{-N\ell}\right|^2\\ &\leq (2\pi)^{2n+2}\max_{k\in\mathbb{N}}\left(k^{2n+2}f_k^2\right)\left(\sum_{\ell\in\mathbb{Z}}|h_\ell|\right)^2 \leq H(2\pi)^{2n+2}\max_{t\geq 0}\phi_n(t)\\ &\leq H(2\pi)^{2n+2}\left(\frac{\beta}{eB_1}(n+1)\right)^{2\beta(n+1)},\end{aligned} \tag{93}$$

where

$$\phi_n(t) := t^{2n+2}\exp\left(-2B_1t^{1/\beta}\right) \tag{94}$$

and H is independent of N and n.

On the other hand, we have

$$\begin{aligned}\|D^{n+1}f;\ L^2\|^2 &= (2\pi)^{2n+2}\sum_{k\in\mathbb{Z}} k^{2n+2}f_k^2 \ge A_1(2\pi)^{2n+2}\sum_{k\in\mathbb{Z}}\phi_n(|k|)\\ &\ge 2A_1(2\pi)^{2n+2}\left(\int_0^\infty \phi_n(t)dt - \max_{t\ge 0}\phi_n(t)\right)\\ &= 2A_1(2\pi)^{2n+2}\left(\int_0^\infty \phi_n(t)dt - \left(\frac{\beta}{eB_1}(n+1)\right)^{2\beta(n+1)}\right) \qquad (95)\end{aligned}$$

using the fact that $\phi_n(t)$, $t \ge 0$, has a single maximum for any n. Further, we estimate

$$\int_0^\infty \phi_n(t)dt = \beta(2B_1)^{-\beta(2n+3)}\Gamma((2n+3)\beta) \ge C\left(\frac{\beta}{eB_1}(n+1)\right)^{2\beta(n+1)} n^{\beta-1/2} \qquad (96)$$

with some $C > 0$, having used

$$\Gamma((2n+3)\beta) \ge c\left(\frac{(2n+3)\beta}{e}\right)^{(2n+3)\beta-1/2} \qquad (97)$$

with some $c > 0$ which is a direct implication of the Stirling formula (11), and then performing further straightforward manipulations.

Comparing finally (93) with (95) and (96), we conclude that

$$\left\langle h_N(x)D^{n+1}f(x)\right\rangle^2 \le cn^{1/2-\beta}\|D^{n+1}f(x);\ L^2\|^2$$

with some $c > 0$, and hence (91) holds with appropriate choice of n_0. ∎

Now we complete the proof of Theorem 5. Let C be a constant from Lemma 6 and let n_0 be as in Lemma 7. Denote

$$\begin{aligned}G_{nN} &:= \|h_N(x)D^{n+1}f(x);\ L^2\|^2 - \left\langle h_N(x)D^{n+1}f(x)\right\rangle^2\\ &= \int_0^1\left(h_N(x)D^{n+1}f(x) - \left\langle h_N(x)D^{n+1}f(x)\right\rangle\right)^2 dx > 0. \qquad (98)\end{aligned}$$

(G_{nN} is strictly positive for any n and N since $h_N(x)D^{n+1}f(x)$ is not a constant: h_N vanishes at some points, but $D^{n+1}f(x)$ is clearly not identically zero.)

By Lemmas 6 and 7, for any N and for any $n > n_0$

$$G_{nN} \ \ge\ \frac{1}{2}C\|D^{n+1}f(x);\ L^2\|^2. \qquad (99)$$

Further, for any $0 < n \le n_0$

$$\lim_{N\to\infty} G_{nN} \ =\ C_1\|D^{n+1}f(x);\ L^2\|^2, \qquad (100)$$

where

$$C_1 := \left\langle\left(h_N - \langle h_N\rangle\right)^2\right\rangle > 0$$

is N-independent positive constant by (89). (A standard way to establish (100) is to subtract from h_N^2 and h_N in (98) their means, represent the resulting zero-mean periodic functions as derivatives of other periodic functions, and then integrate by parts.) It follows from (100) that there exists $C_2 > 0$ such that

$$G_{nN} \geq C_2 \|D^{n+1} f(x);\ L^2\|^2$$

for any N and for any $n \leq n_0$. Combining the latter with (99) implies that

$$G_{nN} \geq C_3 \|D^{n+1} f(x);\ L^2\|^2 \tag{101}$$

for all n and N with $C_3 = \min(C/2, C_2)$.

Next, from (86)–(87)

$$\|\Phi^{\varepsilon,n}(x) - \langle \Phi^{\varepsilon,n}\rangle\,;\ L^2\|^2 = \frac{1}{4}\left(\frac{\varepsilon}{2\pi}\right)^{2n+4} G_{nN} \geq C_4 \left(\frac{\varepsilon}{2\pi}\right)^{2n+4} \|D^{n+1} f(x);\ L^2\|^2. \tag{102}$$

Using again the lower bounds in (95)–(97) implies

$$\|D^{n+1} f(x);\ L^2\|^2 \geq C_5^n n^{2\beta n}$$

with some $C_5 > 0$, which combined with (102) yields

$$\|\Phi^{\varepsilon,n}(x) - \langle \Phi^{\varepsilon,n}\rangle\,;\ L^2\| \geq C_6^n \varepsilon^{n+2} n^{\beta n}. \tag{103}$$

Now, by uniform continuity of L^ε as an operator from H^1 to H^{-1}, we have

$$\|u^{\varepsilon,n} - u^\varepsilon\ ;\ H^1(\mathbb{T})\| \geq \|u^{\varepsilon,n} - \langle u^{\varepsilon,n}\rangle - u^\varepsilon\ ;\ H^1(\mathbb{T})\| \geq C\|R^{\varepsilon,n}(x);\ H^{-1}\|. \tag{104}$$

Combining this with (88) and (103) implies

$$\|u^{\varepsilon,n} - u^\varepsilon\ ;\ H^1(\mathbb{T})\| \geq C_7^n \varepsilon^{n+2} n^{\beta n}. \tag{105}$$

We can now "optimize" the lower bound (105) for any fixed small ε by choosing $n = n(\varepsilon)$ so that the right-hand side of (105) is minimized:

$$\|u^{\varepsilon,n} - u^\varepsilon\ ;\ H^1(\mathbb{T})\| \geq \varepsilon^2 \min_{t\geq 1}\left[(C_7\varepsilon)^t t^{\beta t}\right]. \tag{106}$$

The latter minimum is attained at

$$t = e^{-1}(C_7\varepsilon)^{-1/\beta}; \tag{107}$$

substituting this back into (106), we finally obtain a lower bound of the form

$$\|u^{\varepsilon,n} - u^\varepsilon\ ;\ H^1(\mathbb{T})\| \geq \varepsilon^2 \exp\left[-\beta e^{-1}(C_7\varepsilon)^{-1/\beta}\right]. \tag{108}$$

Finally, since obviously there exists such a positive constant $\tilde{C}_1$ such that $\varepsilon^2 > \tilde{C}_1 \exp(-\varepsilon^{1/\beta})$ for any $0 < \varepsilon \leq 1$, (108) implies (79) for any $0 < \varepsilon \leq 1$, for example, with the above $\tilde{C}_1$ and $\tilde{C}_2 = \beta e^{-1} C_7^{-1/\beta} + 1$. The theorem is proved. ∎

We conclude from Theorem 5 that up to the choice of the constants $C_1, C_2 > 0$, the main error estimate (17) in Theorem 1 is sharp, at least for the above one-dimensional case. Note also that the above lower bound (79) was obtained by "optimizing" the lower bound (105) for a given small ε by choosing $n = n(\varepsilon)$ in the range given by

(107), which is consistent with (16) for the upper bound. In this sense, the range of truncation given by (16) is also "optimal."

Remark 5. Notice that the present one-dimensional case is "integrable" and that Theorem 5 could have been alternatively derived from the exact solution of (69):

$$u^\varepsilon(x) = \int_0^x (F(s) - A^\varepsilon)\, a^{-1}\left(\frac{s}{\varepsilon}\right) ds \; - B^\varepsilon, \tag{109}$$

where $F(x) := -\int_0^x f(t)dt$ is periodic, $A^\varepsilon := \langle F(\cdot)a^{-1}(\cdot/\varepsilon)\rangle\langle a^{-1}\rangle^{-1}$, and $B^\varepsilon := \int_0^1 (F(s){-}A^\varepsilon)a^{-1}(s/\varepsilon)(1{-}s)ds$. One then employs $a^{-1}(x/\varepsilon) = \langle a^{-1}\rangle{+}\varepsilon\langle a^{-1}\rangle \frac{d}{dx}N_1(x/\varepsilon)$ (cf. (80)), in (109) and integrates by parts. Then employing iteratively (81) and integrating by parts n times is expected to explicitly reproduce $u^{\varepsilon,n}$, with the rest being the "error term." The latter would then still have to be analyzed in a fashion similar if not identical to that in the above proofs (the details are omitted).

Remark 6. The same arguments (Lemmas 6 and 7) can be used to obtain lower bounds of finite order in ε if $f \in H^M$ but $f \notin H^{M+1}$ for some $M \in \mathbb{N}$. Namely, on one hand, only a finite number of terms in the asymptotic expansion can be constructed. On the other hand, for each such n (from a finite set) a lower bound of the form (105) holds with appropriate choice of C_7. Optimizing finally with respect to the final set of lower bounds (105), one arrives at an unimprovable polynomial lower bound. On the other hand, if $f \in C^\infty(\mathbb{T})$ and is not from any Gevrey-type class with no other control on the growth of its H^l norms for large l (equivalently, on $\|D^{n+1}f; L^2\|$ for large n), there is no control on the "coefficients" multiplying ε^{n+2} in the error bounds for large n (cf. (104), (86)–(88)), which does not allow us to improve the homogenization in all orders any further. This indicates the importance of the Gevrey extreme regularity of f for the exponential lower bounds.

Remark 7. For analytic $f(x)$ Theorems 1 and 5, i.e., the exponential upper bound (17) and lower bound (79), respectively, both hold with $\beta = 1$. The "rate" of the exponential decay is then determined by the values of the constants C_2 and $\tilde{C}_2$ in (17) and (79). By tracing back the proofs of both theorems, one observes that these constants are dependent on the rate of the exponential decay of the Fourier coefficients of f, i.e., by the constants c_2 in (9) and B_1 in (74). On the other hand, for analytic f, the latter constants are directly related to the "width" of analytic continuation of $f(x)$ into the complex plane off the real axis, i.e., the absolute value of the imaginary part of the "first singularity"; for example, f in (78) has a pole in $x = \pm i/(2\pi)$, corresponding to $B_1 = 1$. In this sense one could argue that the rate of exponential error bound for analytic f is determined by the nearest singularity in the analytic continuation.

Acknowledgment. V. P. Smyshlyaev acknowledges stimulating discussions with Dr. Claudia Wulff of the University of Surrey, Surrey, UK.

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