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MEMORY EFFECT IN HOMOGENIZATION OF A VISCOELASTIC KELVIN–VOIGT MODEL WITH TIME-DEPENDENT COEFFICIENTS

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This paper is motivated by modeling the procedure of formation of a composite material constituted of solid fibers and of a solidifying matrix. The solidification process for the matrix depends on the temperature and on the reticulation rate which thereby influence the mechanical properties of the matrix. The mechanical properties are described by a viscoelastic medium equation of Kelvin−Voigt type with rapidly oscillating periodic coefficients depending on the temperature and the reticulation rate. That is modeled as an initial boundary value problem with time-dependent elasticity and viscosity tensors to account for the solidification, and the mechanical and/or thermal forcing. First we prove the existence and uniqueness of the solution for the problem and obtain *a priori* estimates. Then we derive the homogenized problem, characterize its coefficients including explicit memory terms, and prove that it admits a unique solution. Finally, we prove error bounds for the asymptotic solution, and establish some related regularity properties of the homogenized solution.

Keywords: Composite materials; homogenization; viscoelastic media; memory effect.

AMS Subject Classification: 65M12, 35K55, 35B27, 74A40

1. Introduction

We are motivated by modeling and analyzing the behavior of a composite material made of solid fibers included in a resin (solidifying matrix) which becomes solid when it is heated up (which is known as a reaction of reticulation). We formulate and rigorously analyze a homogenization problem for a relevant viscoelastic initial boundary value problem with time-dependent coefficients, derive homogenized equations and establish error bounds, executing thereby a part of the program announced in Ref. 1. The solidification process depends on the temperature and on the reticulation rate, described by the heat equation coupled to a kinetic equation. For a composite material which has a periodic geometry with a small period $\varepsilon > 0$ corresponding to the distance between neighboring fibers, a homogenization technique applied to this problem in Refs. 20 and 21 reduces it to a homogenized problem with constant coefficients. Certain error bounds have been proved for the difference between the exact solution and the solution of the homogenized problem. In particular, an error bound on the difference between the exact temperature $T_\varepsilon(x, t)$ and the homogenized temperature $T_0(x, t)$ has been obtained.[20,21]

In the present paper, the temperature evaluated at the first stage, as well as the reticulation rate depending on it, are assumed to influence the mechanical properties of the composite material. Namely, the mechanical properties are described by a viscoelastic medium equation (Kelvin−Voigt model) with rapidly oscillating periodic coefficients depending on the temperature and the reticulation rate, that is, on the solution of the above heat transfer problem.[20,21] Specifically, the deformation of a viscoelastic medium under the thermal influence is described by the following problem:

$$\begin{cases} \rho\Big(\dfrac{x}{\varepsilon}\Big)\, \ddot{u}^{\varepsilon}_{T_\varepsilon} - \dfrac{\partial}{\partial x_i}\left(B^{\varepsilon,T_\varepsilon}_{ij}\, \dfrac{\partial \dot{u}^{\varepsilon}_{T_\varepsilon}}{\partial x_j}\right) - \dfrac{\partial}{\partial x_i}\left(A^{\varepsilon,T_\varepsilon}_{ij}\, \dfrac{\partial u^{\varepsilon}_{T_\varepsilon}}{\partial x_j}\right) = f(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0,\tau), \\ u^{\varepsilon}_{T_\varepsilon}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\tau), \\ u^{\varepsilon}_{T_\varepsilon}(x,0) = \dot{u}^{\varepsilon}_{T_\varepsilon}(x,0) = 0, \quad x \in \Omega. \end{cases} \tag{1.1}$$

Here Ω is the physical domain; $u^{\varepsilon}_{T_\varepsilon}(x,t)$ is the unknown displacement vector field, $u^{\varepsilon}_{T_\varepsilon}(x,t) = (u^{\varepsilon}_{T_\varepsilon,k}(x,t))_{1\le k\le n}$; $\dot{u}^{\varepsilon}_{T_\varepsilon}$ and $\ddot{u}^{\varepsilon}_{T_\varepsilon}$ denote the first and the second time derivatives of $u^{\varepsilon}_{T_\varepsilon}$, respectively; the summation over the repeating indices is assumed. The volume density $\rho(\xi)$ is a scalar function which is one-periodic. The smooth vector function $f(x, t)$ is given, and could describe forces due to an external mechanical loading or to thermal effects. The linear elastic tensor $A^{\varepsilon,T_\varepsilon}_{ij}$ and the viscosity tensor $B^{\varepsilon,T_\varepsilon}_{ij}$ are matrix-valued entities:

$$A^{\varepsilon,T_\varepsilon}_{ij} = (A^{kl}_{ij})_{1\le k,l\le n}, \quad B^{\varepsilon,T_\varepsilon}_{ij} = (B^{kl}_{ij})_{1\le k,l\le n}, \qquad i,j = 1,\ldots,n,$$

which are functions of T_ε, where $T_\varepsilon(x, t)$ is the solution of the heat transfer problem,[20,21] and they depend periodically on $\frac{x}{\varepsilon}$.

The viscoelastic properties thereby depend on the temperature: for example, for low temperatures the viscosity tensor B may play more significant role than A while for higher temperatures this may be other way round. Hence this temperature depending Kelvin−Voigt model simulates the thermo-chemo-viscoelastic process in a composite material. In this model first the thermo-chemical problem[20,21] should be solved, and then its solution is used in the viscoelastic equation (1.1). There is a high interest in combined models of this type, see for example a recent paper[2] where a visco-plastic model is considered taking into account a nonlinear hardening effect. Our model (1.1) however takes into account the dependence of the viscoelastic properties on the temperature.

Let us replace T_ε by T_0 (the solution of the homogenized heat problem) in the coefficients A_{ij} and B_{ij} of problem (1.1) and denote by $u^\varepsilon_{T_0}$ a solution of this problem. This new problem with the unknown $u^\varepsilon_{T_0}$ can be regarded as a particular case of a time-dependent problem for the Kelvin–Voigt equation. The elastic and viscous tensors are thereby assumed to depend on the homogenized temperature T_0 (and hence on x and t) in order to reflect the partial solidification of the medium when the temperature decreases. We will hence further consider the following viscoelasticity initial boundary value problem with time-dependent coefficients.

$$\rho\Big(\frac{x}{\varepsilon}\Big)\ddot{u}^\varepsilon - \frac{\partial}{\partial x_i}\left(B_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial \dot{u}^\varepsilon}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(A_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial u^\varepsilon}{\partial x_j}\right) = f(x,t), \tag{1.2}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

$$u^\varepsilon_{|_{t=0}} = \dot{u}^\varepsilon_{|_{t=0}} = 0. \tag{1.4}$$

Here the linear elastic tensor $A_{ij}(x,t,\xi)$ and the viscosity tensor $B_{ij}(x,t,\xi)$ are matrix-valued entities: $A_{ij}(x,t,\xi) = (A^{kl}_{ij}(x,t,\xi))_{1\le k,l\le n}$, $B_{ij}(x,t,\xi) = (B^{kl}_{ij}(x,t,\xi))_{1\le k,l\le n}$, which are periodic with respect to ξ. When the point $\xi = x/\varepsilon$ belongs to a fiber, $A^{kl}_{ij}(x,\frac{x}{\varepsilon},t), B^{kl}_{ij}(x,\frac{x}{\varepsilon},t)$ and $\rho(\frac{x}{\varepsilon})$ are respectively the elasticity, the viscosity and the density coefficients of the fiber, otherwise their values describe the related physical properties of the resin. Since the temperature and the reticulation rate change with x and t, so do A_{ij} and B_{ij}.

A scalar case with time-independent coefficients was studied in Ref. 13 using the Laplace transform methods, with a "fading long-term memory effect" observed. This effect for viscoelasticity (with time-independent coefficients) has been first discovered by Sanchez-Palencia in Ref. 26, Chap. 6. A more detailed result involving the weak convergence of the displacement field of a body to the displacement field of a homogenized material with fading memory has been obtained by Francfort and Suquet in Ref. 16; moreover Ref. 16 studied also homogenization of the viscous dissipation term in the heat equation, although the coefficients were still time-independent.

One of the aims of the present paper is to rigorously establish that the memory effect takes in fact place in a general situation of time-dependent vector problem. In this case the Laplace transform methods cease to be applicable, and we develop instead a version of the method of asymptotic expansions, supplemented by its rigorous justification, including establishing the error bounds. Mathematically, the memory effect is a particular example of more general "nonlocal" effects emerging as a result of homogenization. Various aspects of the latter have been extensively studied and documented in the literature, see e.g. Refs. 3, 4, 6, 8–12, 15, 17, 23b, 24, 27. The present problem nevertheless, from the mathematical point of view, bears essential specifics. Addressing those requires developing certain nontrivial modifications of both the general theory of systems of viscoelasticity type with variable coefficients and of the nonlocal homogenization theory with relevant novel error bounds, addressed in this paper.

One of the main results of the paper is in establishing the structure of the homogenized equation corresponding to (1.2). Namely, we show that under appropriate technical assumptions it has the following form:

$$\widehat{\rho}\ddot{v}(x,t) - \frac{\partial}{\partial x_i}\sigma_i(x,t) = f(x,t), \tag{1.5}$$

where

$$\begin{aligned}\sigma_i(x,t) \;=\; & \widehat{B}_{ij}(x,t)\frac{\partial \dot{v}}{\partial x_j}(x,t) + \hat{\mathrm{A}}_{ij}(x,t)\frac{\partial v}{\partial x_j}(x,t)\\ & + \int_0^t \left(\hat{\mathrm{E}}_{ij}(x,t,t')\frac{\partial \dot{v}}{\partial x_j}(x,t') + \widehat{D}_{ij}(x,t,t')\frac{\partial v}{\partial x_j}(x,t')\right)dt'.\end{aligned} \tag{1.6}$$

Here $\widehat{\rho}$ is the mean value of density ρ, and $\hat{\mathrm{A}}_{ij}$, $\widehat{B}_{ij}$, $\widehat{D}_{ij}$, and $\hat{\mathrm{E}}_{ij}$ are "homogenized" characteristics explicitly expressible in terms of solutions of appropriate "unit cell" problems, see Secs. 3.1 and 3.2. In particular, the "memory" (or "nonlocal") terms are those containing the integro-differential operators with the kernels $\widehat{D}_{ij}$ and $\hat{\mathrm{E}}_{ij}$, explicitly given by (3.27b). The central error bounds on the difference between the exact solution and the homogenized solution are established in Theorem 3.3.

We will adopt following notational conventions throughout the paper. In (1.2) and, henceforth, the summation with respect to repeated indices is implied. A matrix followed by a vector implies a standard multiplication of the matrix by a vector, being a vector; $Q = [0,1]^n$ denotes the reference periodicity cell; $C^\infty_\#(Q)$ stands for the subspace of infinitely smooth functions $C^\infty(\mathbb{R}^n)$ whose elements are periodic with respect to Q, i.e. one-periodic with respect to each of its n variables; $H^1_\#(Q)$ and $L^2_\#(Q)$ denote the closures of $C^\infty_\#(Q)$ in the norms of the standard spaces $H^1(Q)$ and $L^2(Q)$, respectively. For any functional space X, we denote the space X^n by $\mathbf{X}$, and for any tensor $M = M^{kl}_{ij} \in \mathcal{M}_{n^2\times n^2}(\mathbb{R})$ (the set of $n^2 \times n^2$ real matrices), we denote by $\mathcal{I}_M(u,v)_\Omega$ the bilinear form

$$\mathcal{I}_M(u,v)_\Omega = \int_\Omega M^{kl}_{ij}\frac{\partial u_l}{\partial x_j}\frac{\partial v_k}{\partial x_i}dx \quad \forall\, u,v \in \mathbf{H}^1(\Omega).$$

Similarly, we define $\mathcal{I}_M(u,v)_Q$ for all $u,v \in \mathbf{H}^1_\#(Q)$ by the same formula with Ω replaced by Q.

We start by analyzing the general problem (Sec. 2). Namely, we use a version of the Galerkin's method (see e.g. Ref. 14) to prove that for all $\varepsilon > 0$, Eq. (1.2) together with boundary condition (1.3) and general initial conditions specified below, admits a unique solution, and we obtain *a priori* estimates for the solution. Then, in Sec. 3, we develop a modification of the traditional techniques of the method of asymptotic expansions (see, e.g. Refs. 5, 7 and 26) to derive the homogenized equation and to characterize its coefficients. Afterwards we establish that the homogenized problem admits a unique solution v and study the convergence of the exact solution to the approximate solution as ε tends to zero, obtaining tight error bounds on the difference between them, Theorem 3.3. Finally we study related regularity properties of the homogenized solutions (Sec. 4). We derive sufficient conditions for the regularity for a specific case but under rather generic assumptions on the viscosity and elasticity coefficients, allowing them to be microscopically spatially discontinuous as, for example, in a matrix-inclusion composite. We remark in passing that error bounds for elliptic problems with less regular coefficients have recently been obtained, see e.g. Ref. 28.

2. Existence and Uniqueness for Original Problem

Let Ω be a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with a Lipschitz boundary, τ a positive real number, and let $\Omega_\tau := \Omega \times (0, \tau)$. Let for all $i, j = 1, \ldots, n$ and all $x \in \Omega, \xi \in Q$, and $t \in (0, \tau)$ the tensors $A_{ij}(x, t, \xi)$ and $B_{ij}(x, t, \xi)$ belong to $\mathcal{M}_{n,n}(\mathbb{R})$, being measurable functions of their arguments, periodic in ξ. We will set $\xi = x/\varepsilon$ for any positive ε and will regard A_{ij} and B_{ij} as functions on Ω_τ, depending on ε as a parameter, assuming $A_{ij}(x, t, \xi)$ and $B_{ij}(x, t, \xi)$ have sufficient regularity for this to make sense, as will be specified further later in the paper. At the moment we assume that:

(H1) For all $\varepsilon > 0$ and $i, j = 1, \ldots, n$, both A_{ij} and its time derivative $\dot{A}_{ij}$ belong to $L^\infty(\Omega_\tau; \mathcal{M}_{n,n}(\mathbb{R}))$. Moreover, there exists a positive constant ν independent of ε, x and t, such that $\|A_{ij}\|_{L^\infty(\Omega_\tau)} \leq \nu^{-1}$ and $\|\dot{A}_{ij}\|_{L^\infty(\Omega_\tau)} \leq \nu^{-1}$.

(H2) For all $\varepsilon > 0$ the tensors B and $\dot{B}$ belong to $L^\infty(\Omega_\tau; \mathcal{M}_{n^2,n^2,\mathrm{sym}}(\mathbb{R}))$, where $\mathcal{M}_{n^2,n^2,\mathrm{sym}}(\mathbb{R})$ is the set of symmetric (elasticity) tensors such that $B_{ij}^{kl} = B_{kj}^{il} = B_{ji}^{lk}$ and such that $\|\dot{B}_{ij}\|_{L^\infty(\Omega_\tau)} \leq \nu^{-1}$. Additionally, B is uniformly elliptic, i.e. for all symmetric matrices $\eta = (\eta_j^l) \in \mathbb{R}^{n\times n}$, for almost all $(x, t) \in \Omega_\tau$ and all $\varepsilon > 0$

$$\nu \eta_i^k \eta_i^k \leq B_{ij}^{kl}\Big(x, t, \frac{x}{\varepsilon}\Big)\eta_i^k \eta_j^l \leq \nu^{-1}\eta_i^k \eta_i^k.$$

(H3) The Q-periodic function ρ belongs to $L^\infty(Q; \mathbb{R})$ and is uniformly positive, i.e. there exists a constant ρ_1 such that

$$1 \leq \rho(\xi) \leq \rho_1, \quad \text{for all } \xi \in Q.$$

In assumptions (H1) and (H2) we employed the $L^\infty(\Omega_\tau)$-norm of an $n \times n$ matrix-valued function, which can be defined, for example, as the L^∞-norm on Ω_τ for the matrix Euclidean norm.

We consider deformation of a viscoelastic medium with thermal effects, with rapidly oscillating properties in Ω_τ, described by the initial boundary value problem:

$$\rho\Big(\frac{x}{\varepsilon}\Big)\ddot{u}^\varepsilon - \frac{\partial}{\partial x_i}\left(B_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial \dot{u}^\varepsilon}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(A_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial u^\varepsilon}{\partial x_j}\right) = f(x,t), \tag{2.1}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

$$u^\varepsilon_{|_{t=0}} = \varphi, \quad \dot{u}^\varepsilon_{|_{t=0}} = \psi. \tag{2.3}$$

The following theorem establishes the existence, uniqueness and *a priori* estimates for a weak solution to the above initial boundary value problem:

Theorem 2.1. *Let* $f \in L^2(0,\tau;\mathbf{H}^{-1}(\Omega)), \varphi \in \mathbf{H}^1_0(\Omega), \psi \in \mathbf{L}^2(\Omega)$ *and let assumptions* (H1)$-$(H3) *hold. Then, for all* $\varepsilon > 0$, *problem* (2.1)$-$(2.3) *admits a unique weak solution* u^ε *in* $H^1(0,\tau;\mathbf{H}^1_0(\Omega))$, *and there exists a constant* C_1 *depending only on* ν, τ, ρ_1, *and* Ω *such that*

$$\begin{aligned}|||u^\varepsilon|||_{\Omega_\tau} &\equiv \|u^\varepsilon\|_{L^\infty(0,\tau;\mathbf{H}^1_0(\Omega))} + \|\dot{u}^\varepsilon\|_{L^2(0,\tau;\mathbf{H}^1_0(\Omega))} + \|\dot{u}^\varepsilon\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))}\\ &\le C_1(\|f\|_{L^2(0,\tau;\mathbf{H}^{-1}(\Omega))} + \|\varphi\|_{\mathbf{H}^1_0(\Omega)} + \|\psi\|_{\mathbf{L}^2(\Omega)}).\end{aligned} \tag{2.4}$$

Proof. Let $u^\varepsilon(x,t) \in H^1(0,\tau;\mathbf{H}^1_0(\Omega))$ be a weak solution of (2.1)$-$(2.3), i.e.

$$u^\varepsilon(x,0) = \varphi(x), \quad x \in \Omega, \tag{2.5}$$

and, for any $z \in H^1(0,\tau;\mathbf{H}^1_0(\Omega))$ such that $z(x,\tau) = 0$,

$$\begin{aligned}&\int_0^\tau\int_\Omega \left(B_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial \dot{u}^\varepsilon}{\partial x_j} + A_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial u^\varepsilon}{\partial x_j}\right)\cdot\frac{\partial z}{\partial x_i}\,dx\,dt - \int_\Omega \rho\Big(\frac{x}{\varepsilon}\Big)\psi(x)\cdot z(x,0)dx\\ &\quad - \int_0^\tau\int_\Omega \Big(\rho\Big(\frac{x}{\varepsilon}\Big)\dot{u}^\varepsilon(x,t)\cdot\dot{z}(x,t) + f(x,t)\cdot z(x,t)\Big)dx\,dt = 0.\end{aligned} \tag{2.6}$$

(In (2.6) and henceforth we adopt the notational convention of writing the action of $f \in H^{-1}$ on $z \in H^1_0$ as an integral of their product whenever convenient.)

In order to prove the results of the theorem we use the Galerkin's method (see Ref. 14, Sec. 7). We denote by $(w_i)_{i\in\mathbb{N}^*}$ an orthogonal basis of $\mathbf{H}^1_0(\Omega)$, where $\mathbb{N}^*$ stands for set of positive integers. For any fixed $m \in \mathbb{N}^*$ we introduce an approximate problem which consists in finding a function $u^\varepsilon_m(t)$ defined by:

$$u^\varepsilon_m(t) = \sum_{i=1}^m d^i_m(t)w_i(x), \quad 0 \le t \le \tau,$$

where $(d^i_m(t))_{1\le i\le m}$ satisfy the following system of ordinary differential equations: for $j = 1,\ldots,m$

$$(\rho\ddot{u}^\varepsilon_m(t), w_j)_\Omega + \mathcal{I}_B(\dot{u}^\varepsilon_m(t), w_j)_\Omega + \mathcal{I}_A(u^\varepsilon_m(t), w_j)_\Omega = (f(t), w_j)_\Omega, \tag{2.7}$$

$$d^j_m(0) = (\varphi_m, w_j)_\Omega, \quad \dot{d}^j_m(0) = (\psi_m, w_j)_\Omega, \tag{2.8}$$

with φ_m and ψ_m being respectively the orthogonal projections in $\mathbf{H}^1(\Omega)$ of φ and ψ on the finite-dimensional space *Span* $(w_1, w_2, \ldots, w_m)$, with $(\cdot,\cdot)_\Omega$ denoting the standard inner product in $\mathbf{L}^2(\Omega)$ (as well as the action of $\mathbf{H}^{-1}(\Omega)$ on $\mathbf{H}^1_0(\Omega)$).

Using (H3) we conclude that the matrix $(\sqrt{\rho}w_i, \sqrt{\rho}w_j)_{1\le i,j\le m}$ is invertible, so the system (2.7)–(2.8) admits a unique solution u_m^ε in $H^2(0,\tau;\mathbf{H}_0^1(\Omega))$.

We derive next a *priori* estimates for u_m^ε and $\dot{u}_m^\varepsilon$. To this end, we multiply (2.7) by $\dot{d}_m^j(t)$ and sum with respect to $j=1,\dots,m$. Thus we obtain

$$(\rho\ddot{u}_m^\varepsilon(t)_\Omega, \dot{u}_m^\varepsilon(t))_\Omega + \mathcal{I}_B(\dot{u}_m^\varepsilon(t), \dot{u}_m^\varepsilon(t))_\Omega = (f(t), \dot{u}_m^\varepsilon(t))_\Omega - \mathcal{I}_A(u_m^\varepsilon(t), \dot{u}_m^\varepsilon(t))_\Omega.$$

Since $u_m^\varepsilon \in H^2(0,\tau;\mathbf{H}_0^1(\Omega))$, we can re-express the last identity in the following form:

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)}\right) + \mathcal{I}_B(\dot{u}_m^\varepsilon, \dot{u}_m^\varepsilon)_\Omega = (f, \dot{u}_m^\varepsilon)_\Omega - \mathcal{I}_A(u_m^\varepsilon, \dot{u}_m^\varepsilon)_\Omega. \tag{2.9}$$

We aim next at estimating all the terms in the right-hand side of (2.9) in terms of $\mathcal{I}_B(\cdot,\cdot)_\Omega$. We first use the fact that for $u\in\mathbf{H}_0^1(\Omega)$, the assumption (H2) and the standard Korn inequality (see e.g. Refs. 22–24) ensure the equivalence of $\sqrt{\mathcal{I}_B(u,u)_\Omega}$ to the $\mathbf{H}^1$ norm, i.e.

$$\frac{\nu}{2}\|u\|^2_{\mathbf{H}_0^1(\Omega)} \le \mathcal{I}_B(u,u)_\Omega \le n^2\nu^{-1}\|u\|^2_{\mathbf{H}_0^1(\Omega)}. \tag{2.10}$$

Use (H1)–(H2) and the Poincaré–Friedrichs inequality for the terms on the right-hand side of (2.9) and apply (2.10). Thus we check that there exists a constant c_1, e.g. $c_1 = 4\max\{(1+C(\Omega)^2)\nu^{-1}, 2n^4\nu^{-4}\}$ where $C(\Omega)>0$ is the constant appearing in the Poincaré–Friedrichs inequality, such that

$$\frac{d}{dt}\left(\|\sqrt{\rho}\dot{u}_m^\varepsilon\|^2_{\mathbf{L}^2(\Omega)}\right) + \mathcal{I}_B(\dot{u}_m^\varepsilon, \dot{u}_m^\varepsilon)_\Omega \le c_1\Big(\mathcal{I}_B(u_m^\varepsilon, u_m^\varepsilon)_\Omega + \|f\|^2_{\mathbf{H}^{-1}(\Omega)}\Big). \tag{2.11}$$

We deduce from (2.11) that

$$\frac{d}{dt}(\|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)}) \le c_1\Big(\mathcal{I}_B\Big(u_m^\varepsilon(t), u_m^\varepsilon(t)\Big)_\Omega + \|f(t)\|^2_{\mathbf{H}^{-1}(\Omega)}\Big). \tag{2.12}$$

On the other hand, multiplying (2.7) by $d_m^j(t)$ and summing them up for $j=1,\dots,m$, we obtain

$$(\rho\ddot{u}_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega + \mathcal{I}_B(\dot{u}_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega + \mathcal{I}_A(u_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega = (f(t), u_m^\varepsilon(t))_\Omega.$$

By using the symmetry properties of B_{ij} (Assumption H2), we can write the last identity in the following form:

$$\begin{aligned}\frac{d}{dt}\bigg((\rho\dot{u}_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega + \frac{1}{2}\mathcal{I}_B(u_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega\bigg)\\ = \|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\mathcal{I}_{\dot{B}}(u_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega\\ - \mathcal{I}_A(u_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega + (f(t), u_m^\varepsilon(t))_\Omega.\end{aligned}$$

In the same manner as previously, in order to estimate the right-hand side of the last identity via $\mathcal{I}_B(\cdot,\cdot)_\Omega$, we use (H1) and (2.10) to justify that there exists a constant c_2

(e.g. $c_2 = \max\{\frac{1}{2}, 3n^2\nu^{-2} + (1 + C(\Omega)^2)\nu^{-1}\}$), such that

$$\frac{d}{dt}\left((\rho\dot{u}_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega + \frac{1}{2}\mathcal{I}_B(u_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega\right) \\ \leq \|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} + c_2\Big(\mathcal{I}_B\Big(u_m^\varepsilon(t), u_m^\varepsilon(t)\Big)_\Omega + \|f(t)\|^2_{\mathbf{H}^{-1}(\Omega)}\Big).$$

Multiply then the last inequality by a real $\gamma > 0$ and add it to (2.12) to arrive at

$$\frac{dS_\gamma(t)}{dt} \leq c_3\Big(\|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} + \mathcal{I}_B\Big(u_m^\varepsilon(t), u_m^\varepsilon(t)\Big) + \|f(t)\|^2_{\mathbf{H}^{-1}(\Omega)}\Big), \tag{2.13}$$

where, for example, $c_3 = \gamma + (c_1 + \gamma c_2)$ and

$$S_\gamma(t) := \|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\gamma}{2}\mathcal{I}_B(u_m^\varepsilon(t), u_m^\varepsilon(t)) + \gamma(\rho\dot{u}_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega.$$

Lemma 2.1. *For* $0 < \gamma < \frac{\nu}{4\rho_1 C(\Omega)^2}$, *there exists a constant* c_4, *depending only on* ν *and* γ, *such that, for* $0 \leq t \leq \tau$, *the following inequality holds*:

$$\|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\gamma}{2}\mathcal{I}_B(u_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega \leq c_4\left(S_\gamma(0) + \int_0^t \|f\|^2_{\mathbf{H}^{-1}(\Omega)}dt\right). \tag{2.14}$$

Proof. By using assumptions (H1) and (H3), the Poincaré−Friedrichs inequality and (2.10), it is easy to see that for $0 < \gamma \leq \nu/(4\rho_1 C(\Omega)^2)$ we have

$$\frac{1}{2}\|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\gamma}{4}\mathcal{I}_B(u_m^\varepsilon(t), u_m^\varepsilon(t)) - |\gamma(\rho\dot{u}_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega| \geq 0.$$

Thus we deduce that

$$S_\gamma(t) \geq \frac{1}{2}\|\sqrt{\rho}\dot{u}_m^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\gamma}{4}\mathcal{I}_B(u_m^\varepsilon(t), u_m^\varepsilon(t))_\Omega. \tag{2.15}$$

Furthermore, we fix γ in $]0, \frac{\nu}{4\rho_1 C(\Omega)^2}[$ in inequality (2.13) and, using (2.15), find a constant $c_5 = c_3\max\{\frac{4}{\gamma}, 2\}$ such that

$$\frac{dS_\gamma(t)}{dt} \leq c_5\Big(S_\gamma(t) + \|f(t)\|^2_{\mathbf{H}^{-1}(\Omega)}\Big).$$

Then, by the Gronwall's inequality (e.g. Ref. 14, Sec. B2.j), we deduce that

$$S_\gamma(t) \leq e^{c_5 t}\left(S_\gamma(0) + \int_0^t \|f(t)\|^2_{\mathbf{H}^{-1}(\Omega)}dt\right), \quad t \in [0, \tau].$$

Finally, comparing the last inequality with (2.15), we obtain (2.14) with e.g. $c_4 = 2e^{c_5\tau}$. □

Now we will use inequality (2.14) to establish *a priori* estimates for u_m^ε. Indeed, we estimate the term on the left of (2.14) from below by using (2.10) and thus we check that for all $t \in [0, \tau]$ we have

$$\|u_m^\varepsilon\|_{\mathbf{H}_0^1(\Omega)} + \|\dot{u}_m^\varepsilon\|_{\mathbf{L}^2(\Omega)} \leq c_6\Big(\|f\|_{L^2(0,\tau;\mathbf{H}^{-1}(\Omega))} + \|\varphi\|_{\mathbf{H}_0^1(\Omega)} + \|\psi\|_{\mathbf{L}^2(\Omega)}\Big). \tag{2.16}$$

On the other hand, by integrating (2.11) with respect to t and by using (2.16), we verify the following estimate:

$$\|\dot{u}_m^\varepsilon\|^2_{L^2(0,\tau;\mathbf{H}_0^1(\Omega_\tau))} \le c_7\Big(\|f\|^2_{L^2(0,\tau;\mathbf{H}^{-1}(\Omega))} + \|\varphi\|^2_{\mathbf{H}_0^1(\Omega)} + \|\psi\|^2_{\mathbf{L}^2(\Omega)}\Big). \tag{2.17}$$

From estimates (2.16) and (2.17), we conclude that the sequences $(u_m^\varepsilon)_{m\in\mathbb{N}^*}$ and $(\dot{u}_m^\varepsilon)_{m\in\mathbb{N}^*}$ are bounded in $L^\infty(0,\tau;\mathbf{H}_0^1(\Omega))$ and $L^2(0,\tau;\mathbf{H}_0^1(\Omega))\cap L^\infty(0,\tau;\mathbf{L}^2(\Omega))$, respectively, uniformly in m (and ε). Therefore we can extract a subsequence $(u_{m'}^\varepsilon)_{m'\in\mathbb{N}^*}$ such that, when $m'\to\infty$, we have:

$$u_{m'}^\varepsilon \overset{*}{\rightharpoonup} u^\varepsilon \in L^\infty(0,\tau;\mathbf{H}_0^1(\Omega)) \quad \text{and} \quad \dot{u}_{m'}^\varepsilon \overset{*}{\rightharpoonup} \dot{u}^\varepsilon \in L^2(0,\tau;\mathbf{H}_0^1(\Omega))\cap L^\infty(0,\tau;\mathbf{L}^2(\Omega)),$$

where $\overset{*}{\rightharpoonup}$ denotes the weak-star convergence. Lastly, we integrate identity (2.7) over $[0,\tau]$ (integrating the first term once by parts in t) and pass to the limit as $m'\to\infty$ using the above convergence results. So we prove that u^ε satisfy (2.5) and (2.6). In the same way we pass to the limit in estimates (2.16) and (2.17) and prove that u^ε satisfy the estimate (2.4) of the theorem. Thus we have proved that there exists a function u^ε belonging to $H^1(0,\tau;\mathbf{H}_0^1(\Omega))$ and satisfying the weak formulation of problem (2.1) and (2.3).

In order to prove that the above solution is unique, it suffices to show that problem (2.5) and (2.6) with $f\equiv\varphi\equiv\psi\equiv 0$ has no nontrivial solutions. To verify this, we fix $0\le s\le\tau$ and substitute

$$v(x,t) = \begin{cases} \displaystyle\int_t^s u^\varepsilon(x,r)dr, & \text{when } 0\le t\le s,\\ 0, & \text{when } s\le t\le\tau, \end{cases}$$

into the identity (2.6), adopting in this case the form

$$-\int_0^\tau\int_\Omega \rho\dot{u}^\varepsilon\dot{v}dx\,dt + \int_0^\tau \mathcal{I}_B(\dot{u}^\varepsilon,v)_\Omega dt + \int_0^\tau \mathcal{I}_A(u^\varepsilon,v)_\Omega dt = 0, \quad u^\varepsilon(x,0)=0.$$

After integration by parts in the first and the second terms, we obtain

$$\frac{1}{2}\|\sqrt{\rho}u^\varepsilon(s)\|^2_{L^2(\Omega)} + \int_0^s \mathcal{I}_B(u^\varepsilon,u^\varepsilon)dt \le \int_0^s \Big(|\mathcal{I}_{\dot{B}}(u^\varepsilon,v)_\Omega| + |\mathcal{I}_A(u^\varepsilon,v)_\Omega|\Big)dt. \tag{2.18}$$

Now, by using (2.10), we estimate the right-hand side of (2.18) via $\mathcal{I}_B(\cdot,\cdot)_\Omega$ and obtain inequality

$$\frac{1}{2}\|\sqrt{\rho}u^\varepsilon(s)\|^2_{\mathbf{L}^2(\Omega)} + \left(1-\frac{4n^2s}{\nu^2}\right)\int_0^s \mathcal{I}_B(u^\varepsilon,u^\varepsilon)_\Omega dt \le 0.$$

Finally, we choose $\tau=\tau_1$ small enough so that $\tau_1<\frac{\nu^2}{4n^2}$. The latter estimate implies that for any s in $[0,\tau_1]$, we have $\|u^\varepsilon(s)\|_{\mathbf{L}^2(\Omega)}=0$. Finally, if $\tau_1<\tau$, we apply repeatedly the same argument on the smaller intervals within $[0,\tau]$ to deduce that $u^\varepsilon\equiv 0$. □

3. Asymptotic Expansion of the Solution

In this section, we consider the above model of a viscoelastic deformation of an ε-periodic composite material, treating now ε as a small parameter. The functions $\rho(\xi)$, $A_{ij}(x,t,\xi)$, and $B_{ij}(x,t,\xi)$ are Q-periodic with respect to ξ (e.g. with the unit periodicity cell Q). We assume that these functions satisfy hypotheses fully analogous to (H1)−(H3), as clarified later. We describe the asymptotic behavior of the solution of problem (1.2)−(1.4) when ε is small.

According to the traditional asymptotic expansion method, a formal asymptotic solution to the problem (2.1)−(2.3) is sought in the following two-scale form:

$$u^\varepsilon(x,t) \sim v(x,t) + \varepsilon N\Big(x,t,\frac{x}{\varepsilon}\Big) + \varepsilon^2 u^{(2)}\Big(x,t,\frac{x}{\varepsilon}\Big) + \cdots. \tag{3.1}$$

Here $v(x,t)$ is the leading term, $N(x,t,\frac{x}{\varepsilon})$ is the "corrector"-term found from appropriate "unit cell" problem, and $N(x,t,\xi), u^{(2)}(x,t,\xi)$ are assumed to be Q-periodic in ξ. Near the boundary $\partial\Omega$ the asymptotic expansion (3.1) is expected to be supplemented by a usual "boundary layer"(cf. e.g. Ref. 5, Chap. 9).

Substituting the ansatz (3.1) into (2.1) and collecting formally the terms with equal powers of ε, we obtain a sequence of initial boundary value problems which will be stated explicitly later on. A key role in the subsequent asymptotic constructions will be played by various versions of the following "cell" problem, which we first study here in some generality. For any $x \in \overline{\Omega}$, we seek $u(x,\cdot,\cdot) \in H^1(0,\tau;\mathcal{H})$ such that

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial \dot{u}}{\partial \xi_j}\right) - \frac{\partial}{\partial \xi_i}\left(A_{ij}(x,t,\xi)\frac{\partial u}{\partial \xi_j}\right) = g(x,t,\xi), \tag{3.2}$$

$$u(x,0,\xi) = \phi(x,\xi). \tag{3.3}$$

Here $\mathcal{H}$ is Hilbert space defined by

$$\mathcal{H} := \{u \in \mathbf{H}^1_\#(Q); \langle u(\xi)\rangle_\xi = 0\}, \tag{3.4}$$

equipped with the norm

$$\|u\|_{\mathcal{H}} := \|\nabla_\xi u\|_{(\mathbf{L}^2(Q))^n}.$$

Henceforth, the angular brackets with subscript ξ denote the mean of the appropriate function with respect to ξ:

$$\langle u(\xi)\rangle_\xi := \int_Q u(\xi)d\xi. \tag{3.5}$$

We will also make use of space $\mathbf{H}^{-1}_\#(Q)$ which is by definition a (dual) space of linear continuous functionals g on $\mathbf{H}^1_\#(Q)$, with their values upon action on $u \in \mathbf{H}^1_\#(Q)$ denoted $\langle g,u\rangle$. For $g \in \mathbf{H}^{-1}_\#(Q)$, $\langle g\rangle_\xi$ is defined as $\langle g,\mathbf{1}\rangle$, where $\mathbf{1}$ is the identical unity for all the components, with the latter definition being consistent with (3.5) for more regular g, e.g. $g \in \mathbf{L}^2(Q)$.

Theorem 3.1. *Assume that for all* $x \in \overline{\Omega}$ *the functions* ϕ *and* g *belong respectively to* $\mathcal{H}$ *and* $L^2(0,\tau;\mathbf{H}^{-1}_{\#}(Q))$. *Let* $g(x,t,\cdot)$ *satisfy* $\langle g\rangle_\xi = 0$. *Let also for any* $x \in \Omega$ *the straightforward modifications of assumptions* (H1)–(H2) *hold, namely with* x *replaced by* ξ, Ω *by* Q, *and* Ω_τ *by* $Q_\tau := Q \times (0,\tau)$. *Then the problem* (3.2)–(3.3) *admits for any* x *a unique* (*weak*) *solution* $u(x,\cdot,\cdot)$ *in* $H^1(0,\tau;\mathcal{H})$. *Moreover, there exists a constant* C_2 *which depends only on* ν *and* τ *such that, for any* x *in* $\overline{\Omega}$,

$$\|u\|_{L^\infty(0,\tau;\mathcal{H})} + \|\dot{u}\|_{L^2(0,\tau;\mathcal{H})} \le C_2(\|g\|_{L^2(0,\tau;\mathbf{H}^{-1}(Q))} + \|\phi\|_{\mathcal{H}}). \tag{3.6}$$

The proof of Theorem 2.1 is fully analogous to that of Theorem 1.1. Namely, for the existence we use the Galerkin approximations techniques in $\mathcal{H}$ (exploiting the invertibility of the matrix analogous in the new variables to $\mathcal{I}_B(w_i, w_j)$), and the uniqueness follows immediately from a straightforward modification of the Gronwall's inequality. We do not reproduce the proof here.

We next study regularity properties of the solutions to the problem (3.2)–(3.3). Mathematically, the latter is subsequently required for constructing the "higher-order" terms in the asymptotic expansion and for eventually obtaining the error bounds. Physically, the dependence of the viscoelastic coefficients A_{ij} and B_{ij} on x and t results from the dependence of these coefficients on the temperature. Consequently, the regularity of the temperature in x and t studied e.g. in Ref. 7 determines the regularity of A_{ij} and B_{ij} on x and t, which allows us to study the regularity of the solution of problem (3.2) and (3.3). With this aim we state and proof the following lemma.

Lemma 3.1. *Let* $u(x,\cdot,\cdot) \in H^1(0,\tau;\mathcal{H})$ *be the unique solution of problem* (3.2) *and* (3.3) *and assume that* $A_{ij}, B_{ij} \in C^p(\overline{\Omega}_\tau; L^\infty(Q)), \phi \in C^p(\overline{\Omega};\mathcal{H})$ *and* $g \in C^p(\overline{\Omega}_\tau;\mathbf{H}^{-1}(Q))$, *for some* $p \ge 1$. *Then we have*

$$u, \dot{u} \in C^p(\overline{\Omega}_\tau;\mathcal{H}).$$

Proof. The proof is by induction in p, i.e. via a version of a "bootstrap" argument. Theorem 3.1 implies that for any x in $\overline{\Omega}$ there exists a unique solution $u(x,\cdot,\cdot)$ in $H^1([0,\tau];\mathcal{H})$ and it satisfies (3.6). It is easy to see first that $u(x,\cdot,\cdot)$ is in fact in $C([0,\tau];\mathcal{H})$, cf. Ref. 14, Sec. 5.9.2. For example, fixing x and introducing a small h,

$$\begin{aligned}\|u(x,t+h,\cdot) - u(x,t,\cdot)\|_{\mathcal{H}} &= \left\|\int_t^{t+h} \dot{u}(x,s,\cdot)ds\right\|_{\mathcal{H}} \\ &\le \left(\int_t^{t+h} \|\dot{u}(x,s,\cdot)\|^2_{\mathcal{H}} ds\right)^{1/2} h^{1/2} \\ &\le ch^{1/2} \to 0 \quad \text{as } h \to 0\end{aligned}$$

with appropriate constant c.

Let next the assumptions of the lemma hold for $p = 1$ and let us prove that then $u, \dot{u} \in C^1(\overline{\Omega}_\tau;\mathcal{H})$. Initially we will use only the fact that $\phi \in C(\overline{\Omega};\mathcal{H})$ and $g \in C(\overline{\Omega}_\tau;\mathbf{H}^{-1}(Q))$.

For any fixed value of x in $\overline{\Omega}$ and for any vector $h \in \mathbb{R}^n$ such that $x+h \in \overline{\Omega}$, we have $u(x+h,\cdot,\cdot) \in C([0,\tau];\mathcal{H})$. We subtract Eq. (3.2) written down for x from the same equation with x replaced by $x+h$ and integrate the result over $Q \times [0,\tau]$. Thus we verify that for any $z \in H^1(0,\tau;\mathbf{H}^1_{\#}(Q))$, the function u^h defined by

$$u^h(t,\xi) := u(x+h,t,\xi) - u(x,t,\xi), \quad (t,\xi) \in [0,\tau] \times Q$$

satisfies

$$\int_0^\tau \int_Q A_{ij} \frac{\partial u^h}{\partial \xi_j} \cdot \frac{\partial z}{\partial \xi_i} d\xi dt + \int_0^\tau \int_Q B_{ij} \frac{\partial \dot{u}^h}{\partial \xi_j} \cdot \frac{\partial z}{\partial \xi_i} d\xi dt = \int_0^\tau \int_Q g^h \cdot z \, d\xi dt,$$
$$u^h_{|_{t=0}} = \phi^h,$$

where $\phi^h(x,\xi) = \phi(x+h,\xi) - \phi(x,\xi)$, and

$$\begin{aligned} g^h(x,t,\xi) \;=\; & g(x+h,t,\xi) + \frac{\partial}{\partial \xi_i}\left((A_{ij}(x+h,t,\xi) - A_{ij}(x,t,\xi)) \frac{\partial u}{\partial \xi_j}(x+h,t,\xi) \right) \\ & - g(x,t,\xi) + \frac{\partial}{\partial \xi_i}\left((B_{ij}(x+h,t,\xi) - B_{ij}(x,t,\xi)) \frac{\partial \dot{u}}{\partial \xi_j}(x+h,t,\xi) \right). \end{aligned}$$

From (3.6) we obtain

$$\|u^h\|_{L^\infty(0,\tau;\mathcal{H})} \le C_2(\|\varphi^h\|_{\mathcal{H}} + \|g^h\|_{L^2(0,\tau;\mathbf{H}^{-1}(Q))}).$$

The continuity of functions A_{ij}, B_{ij}, φ, and g on $\overline{\Omega}_\tau$ implies that the term on the right-hand side of the last inequality tends to zero as h vanishes, and consequently, $\|u^h\|_{\mathcal{H}}$ converges to zero as $h \to 0$ for any t in $[0,\tau]$. Then, using the fact that $u \in C([0,\tau];\mathcal{H})$ for any x in $\overline{\Omega}$, we deduce that, for any $(x,t) \in \overline{\Omega}_\tau$, the following relation holds:

$$\|u(x+h,t+\zeta,\xi) - u(x,t,\xi)\|_{\mathcal{H}} \to 0, \quad \text{when } (h,\zeta) \to (0,0) \text{ in } \mathbb{R}^{n+1}.$$

So $u \in C(\overline{\Omega}_\tau;\mathcal{H})$ and now we will prove that $\dot{u}$ belongs to $C(\overline{\Omega}_\tau;\mathcal{H})$. To this end, we fix (x,t) in $\overline{\Omega}_\tau$ and write (3.2) in the form:

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi) \frac{\partial \dot{u}}{\partial \xi_j} \right) = \mathcal{G}(x,t,\xi).$$

Here,

$$\mathcal{G}(x,t,\xi) := \frac{\partial}{\partial \xi_i}\left(A_{ij}(x,t,\xi) \frac{\partial u}{\partial \xi_j} \right) + g(x,t,\xi).$$

By using (H1) and (H2) and the Korn inequality which is valid for any u in $\mathbf{H}^1(Q)$ we conclude that the $\mathcal{H}$-norm and $\sqrt{\mathcal{I}_B(\cdot,\cdot)_Q}$ are equivalent. Therefore, there exists a constant c depending only on ν and the space dimension n, such that

$$\|\dot{u}\|_{\mathcal{H}} \le c\|\mathcal{G}\|_{\mathbf{H}^{-1}(Q)}.$$

Since the last inequality holds for all $(x,t) \in \overline{\Omega}_\tau$, we use the same techniques as above to prove that for small parameters $(h,\zeta) \in \mathbb{R}^{n+1}$, such that $(x+h,t+\zeta)$ in $\overline{\Omega}_\tau$, we have

$$\|\dot{u}(x+h,t+\zeta,\xi) - \dot{u}(x,t,\xi)\|_{\mathcal{H}} \leq c\|\mathcal{G}(x+h,t+\zeta,\xi) - \mathcal{G}(x,t,\xi)\|_{\mathbf{H}^{-1}(Q)}.$$

Taking into account that u, g, A_{ij}, and B_{ij} are continuous on $\overline{\Omega}_\tau$ one can show that $\mathcal{G}$ belongs to $C(\Omega_\tau; \mathbf{H}^{-1}(Q))$ and so $\dot{u} \in C(\overline{\Omega}_\tau; \mathcal{H})$. Thus we have proved that for a given $g \in C(\overline{\Omega}_\tau; \mathbf{H}^{-1}(Q))$ and $\phi \in C(\overline{\Omega}; \mathcal{H})$, the function u, solution of problem (3.2) and (3.3), and its time derivative $\dot{u}$ both belong to $C(\overline{\Omega}_\tau; \mathcal{H})$.

Next formally differentiate Eq. (3.2) with respect to t and use the fact that functions $A_{ij}, B_{ij} \in C^1(\overline{\Omega}_\tau; L^\infty(Q)), \phi \in C^1(\overline{\Omega}; \mathcal{H})$, and $g \in C^1(\overline{\Omega}_\tau; \mathbf{H}^{-1}(Q))$ to verify that $\dot{u}$ satisfies the same type of problem as (3.2) and (3.3) with the right-hand side belonging to $C(\overline{\Omega}_\tau; \mathbf{H}^{-1}(Q))$ and the initial condition $\dot{u}(0)$ in $C(\overline{\Omega}; \mathcal{H})$. Therefore the above result implies that the second time derivative of u denoted $\ddot{u}$ is in fact in $C(\overline{\Omega}_\tau; \mathcal{H})$.

Now we will study the differentiability of u with respect to x. To this end, we fix a positive integer $l \leq n$ and differentiate formally Eq. (3.2) with respect to x_l. As above, we check that there exists a function $\bar{u}$ such that $\bar{u}, \dot{\bar{u}} \in C(\overline{\Omega}_\tau; \mathcal{H})$, satisfying the following:

$$\int_0^\tau \int_Q \left(B_{ij} \frac{\partial \dot{\bar{u}}}{\partial \xi_j} + A_{ij} \frac{\partial \bar{u}}{\partial \xi_j} \right) \cdot \frac{\partial z}{\partial \xi_i} d\xi dt = \int_0^\tau \int_Q \bar{g} \cdot z \, d\xi dt$$

$$\bar{u}(x,0,\xi) = \frac{\partial \phi}{\partial x_l}(x,\xi)$$

with $\bar{g} = \frac{\partial g}{\partial x_l} + \frac{\partial}{\partial \xi_i}\left(\frac{\partial B_{ij}}{\partial x_l} \frac{\partial \dot{u}}{\partial \xi_j}\right) + \frac{\partial}{\partial \xi_i}\left(\frac{\partial A_{ij}}{\partial x_l} \frac{\partial u}{\partial \xi_j}\right)$.

By using (3.6) and the same techniques as above, we check that for a small scalar parameter δ and for all $x \in \overline{\Omega}$, we have

$$\|\tilde{u} - \bar{u}\|_{L^\infty(0,\tau;\mathcal{H})} \leq C_2 \Big(\|(\tilde{u} - \bar{u})(0)\|_{\mathcal{H}} + \|g_\delta\|_{L^2(0,\tau;\mathbf{H}^{-1}(Q))} \Big),$$

where $\tilde{u}(x,t,\xi) = \frac{u(x+\delta e_l,t,\xi) - u(x,t,\xi)}{\delta}$; $(e_l)_{1 \leq l \leq n}$ is the canonical basis of $\mathbb{R}^n$, and

$$\begin{aligned} g_\delta(x) &= \frac{\partial}{\partial \xi_i}\left(\frac{B_{ij}(x+\delta e_l,t,\xi) - B_{ij}(x,t,\xi)}{\delta} \frac{\partial \dot{u}}{\partial \xi_j}(x+\delta e_l,t,\xi) \right) - \bar{g}(x,t,\xi) \\ &\quad + \frac{\partial}{\partial \xi_i}\left(\frac{A_{ij}(x+\delta e_l,t,\xi) - A_{ij}(x,t,\xi)}{\delta} \frac{\partial u}{\partial \xi_j}(x+\delta e_l,t,\xi) \right) \\ &\quad + \frac{g(x+\delta e_l,t,\xi) - g(x,t,\xi)}{\delta}. \end{aligned}$$

Then, taking into account that u and $\dot{u}$ belong to $C(\overline{\Omega}_\tau; \mathcal{H})$ and using the assumption of the lemma (for $p = 1$), we prove that $\frac{\partial u}{\partial x_l}$ exists and is equal to $\bar{u}$. Thus we have $\frac{\partial u}{\partial x_l}, \frac{\partial \dot{u}}{\partial x_l} \in C(\overline{\Omega}_\tau; \mathcal{H})$ and thereby complete the proof of the assertion of the lemma for $p = 1$.

The remainder of the proof repeats the above argument with minor modifications. Namely, assume that the statement of the lemma is valid for some positive integer p and suppose that $\phi \in C^{p+1}(\overline{\Omega};\mathcal{H}), g \in C^{p+1}(\overline{\Omega}_\tau;\mathbf{H}^{-1}(Q))$, and $A_{ij}, B_{ij} \in C^{p+1}(\overline{\Omega}_\tau; L^\infty(Q))$. Then the differentiation of (3.2) with respect to t implies that $\dot{u}$ satisfies (3.2) with the right-hand side in $C^p(\overline{\Omega}_\tau;\mathbf{H}^{-1}(Q))$. Similarly, differentiating (3.2) with respect to $x_l, l = 1,\dots,n$, we check that $\frac{\partial u}{\partial x_l}$ is a solution of (3.2) with the right-hand side belonging to $C^p(\overline{\Omega}_\tau;\mathbf{H}^{-1}(Q))$. Thus, the induction assumption implies that $\dot{u}, \ddot{u}, \frac{\partial u}{\partial x_l}, \frac{\partial \dot{u}}{\partial x_l} \in C^p(\overline{\Omega}_\tau;\mathcal{H})$ and we deduce the result of the lemma for $p+1$, completing the proof. □

3.1. *The main unit cell problem*

Substituting the ansatz (3.1) into (2.1) and taking into account that v is independent of ξ, the identification of the terms corresponding to order ε^{-1} leads to the following equation:

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial^2 N}{\partial \xi_j \partial t}(x,t,\xi)\right) - \frac{\partial}{\partial \xi_i}\left(A_{ij}(x,t,\xi)\frac{\partial N}{\partial \xi_j}(x,t,\xi)\right) = F(x,t,\xi), \quad (3.7)$$

where

$$F(x,t,\xi) := \frac{\partial}{\partial \xi_i} B_{ik}(x,t,\xi)\frac{\partial \dot{v}}{\partial x_k}(x,t) + \frac{\partial}{\partial \xi_i} A_{ik}(x,t,\xi)\frac{\partial v}{\partial x_k}(x,t). \quad (3.8)$$

Equation (3.7) has to be supplemented by the initial condition

$$N(x,0,\xi) = 0, \quad (3.9)$$

to comply with the first initial condition in (2.3). This leads to a version of the main "unit cell" problem, typical for homogenization problems. By Theorem 3.1 its solution $N(x,t,\xi)$, periodic in ξ, exists for any given $v(x,t)$ such that $\partial v/\partial x_k(x,\cdot) \in H^1(0,\tau)$ for all $x \in \Omega$ and $k = 1,\dots,n$. The solution $N(x,t,\xi)$ exists and is unique up to a function depending only on x and t (i.e. a constant with respect to ξ). To select a unique solution, we require that N has zero mean value with respect to ξ. Henceforth, we apply this selection criterium whenever a boundary value problem with periodic conditions is stated, cf. Theorem 3.1. The following lemma establishes the structure of function N.

Lemma 3.2. *The following representation holds*:

$$N(x,t,\xi) = \int_0^t \mathcal{N}_k^B(x,t-t',t',\xi)\frac{\partial \dot{v}}{\partial x_k}(x,t')dt' + \int_0^t \mathcal{N}_k^A(x,t-t',t',\xi)\frac{\partial v}{\partial x_k}(x,t')dt'. \quad (3.10)$$

Here $\mathcal{N}_k^A(x,t,s,\xi)$ and $\mathcal{N}_k^B(x,t,s,\xi)$ are periodic with respect to ξ solutions of the following initial boundary value problems:

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,s+t,\xi)\frac{\partial \dot{\mathcal{N}}_k^A}{\partial \xi_j}(x,t,s,\xi)\right) - \frac{\partial}{\partial \xi_i}\left(A_{ij}(x,s+t,\xi)\frac{\partial \mathcal{N}_k^A}{\partial \xi_j}(x,t,s,\xi)\right) = 0, \quad (3.11)$$

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,s+t,\xi)\frac{\partial \dot{\mathcal{N}}_k^B}{\partial \xi_j}(x,t,s,\xi)\right)-\frac{\partial}{\partial \xi_i}\left(A_{ij}(x,s+t,\xi)\frac{\partial \mathcal{N}_k^B}{\partial \xi_j}(x,t,s,\xi)\right)=0, \tag{3.12}$$

$$\mathcal{N}_k^A(x,0,s,\xi)=g_k^A(x,s,\xi),\quad \mathcal{N}_k^B(x,0,s,\xi)=g_k^B(x,s,\xi). \tag{3.13}$$

In turn, $g_k^A(x,s,\xi), g_k^B(x,s,\xi)$ *solve the following cell problems*:

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,s,\xi)\frac{\partial}{\partial \xi_j}g_k^A(x,s,\xi)\right)=\frac{\partial}{\partial \xi_i}A_{ik}(x,s,\xi), \tag{3.14}$$

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,s,\xi)\frac{\partial}{\partial \xi_j}g_k^B(x,s,\xi)\right)=\frac{\partial}{\partial \xi_i}B_{ik}(x,s,\xi). \tag{3.15}$$

Proof. We check here the "formal" part of the proof of Lemma 2.2. The existence and uniqueness of the solutions of these boundary value problems will be established later. From (3.10), we obtain

$$\begin{aligned}\frac{\partial N}{\partial t} &= \int_0^t \frac{\partial}{\partial t}\mathcal{N}_k^B(x,t-t',t',\xi)\frac{\partial \dot{v}}{\partial x_k}(x,t')dt' \\ &\quad + \int_0^t \frac{\partial}{\partial t}\mathcal{N}_k^A(x,t-t',t',\xi)\frac{\partial v}{\partial x_k}(x,t')dt' \\ &\quad + \mathcal{N}_k^B(x,0,t,\xi)\frac{\partial \dot{v}}{\partial x_k}(x,t)+\mathcal{N}_k^A(x,0,t,\xi)\frac{\partial v}{\partial x_k}(x,t).\end{aligned} \tag{3.16}$$

Substitute (3.10) and (3.16) into (3.7). Using (3.11)–(3.13) we conclude that the left-hand side of (3.7) equals

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial}{\partial \xi_j}\,g_k^B\right)\frac{\partial \dot{v}}{\partial x_k}(x,t)-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial}{\partial \xi_j}\,g_k^A\right)\frac{\partial v}{\partial x_k}(x,t). \tag{3.17}$$

Finally, using (3.14) and (3.15), we conclude that (3.17) transforms into the right-hand side of (3.7) given by (3.8). Obviously (3.10) satisfies also (3.9). □

Notice that according to (3.15), for any fixed x and t, g_k^B are "standard" unit cell solutions corresponding to the elliptic operator with coefficients $B_{ij}(x,t,\xi)$ in homogenization for the "classical" elliptic operators with periodic coefficients (cf. e.g. Refs. 5 and 29), whereas g_k^A does not have such a direct "standard" analogue.

Lemma 3.3. (i) *Let the modification of assumptions* (H1) *and* (H2) *stated in Theorem* 3.1 *hold. Then for any* $k=1,\dots,n$, *Eq.* (3.14) *admits a unique solution such that for all* $(x,t)\in\Omega_\tau$, $g_k^A\in\mathcal{H}^n$. *Moreover, there exists a constant* C_3 *which depends only on* ν *and* n *such that*

$$\|g_k^A\|_{\mathcal{H}^n}+\|\dot{g}_k^A\|_{\mathcal{H}^n}\le C_3,\quad \forall\,(x,t)\in\Omega_\tau. \tag{3.18}$$

(ii) *Let* $p \in \mathbb{N}^*$ *and* $A_{ij}, B_{ij} \in C^p(\overline{\Omega}_\tau; L^\infty(Q))$. *Then, for any* $k = 1, \ldots, n$,

$$g_k^A \in C^p(\overline{\Omega}_\tau; \mathcal{H}^n). \tag{3.19}$$

Proof. We fix k and consider x and s as parameters, i.e. g_k^A depends only on ξ.

(i) Let $g_k^A(\xi)$ be a weak solution of (3.14), i.e.

$$\int_Q \frac{\partial \phi^*}{\partial \xi_i} \cdot B_{ij} \frac{\partial g_k^A}{\partial \xi_j} d\xi = -\int_Q \frac{\partial \phi^*}{\partial \xi_i} \cdot A_{ik} d\xi, \quad \forall\, \phi \in (\mathbf{H}^1(Q))^n. \tag{3.20}$$

Make use of the equivalence between $\sqrt{\mathcal{I}_B(\cdot,\cdot)_Q}$ and the $\mathcal{H}$-norm mentioned before. We check that the term on the left-hand side of (3.20) is a symmetric and positive definite bilinear form on $\mathcal{H}^n$, while the right-hand side is a continuous linear form on $\mathcal{H}^n$. Thus, the Lax–Milgram lemma implies the existence and uniqueness of the solution $g_k^A \in \mathcal{H}^n$ of (3.20) and we obtain the following estimate:

$$\|g_k^A\|_{\mathcal{H}^n} \leq 2n\nu^{-2}. \tag{3.21}$$

In order to study the existence of the time derivative of g_k^A, we use the same argument as in the proof of Lemma 3.1 to conclude that $\dot{g}_k^A \in \mathcal{H}^n$ exists and is bounded in the $\mathcal{H}^n$-norm by a constant depending only on ν. Thus we arrive at (3.18).

(ii) The proof of (3.19) is by induction in p and is similar to that in Lemma 3.1. First, we prove that the continuity of A_{ij} on $\overline{\Omega}_\tau$ implies the continuity of g_k^A in x and t. Second, we check that if $A_{ij} \in C^{p+1}(\overline{\Omega}_\tau; L^\infty(Q))$, the function $\frac{\partial g_k^A}{\partial x_l'}$ exists in $\mathcal{H}^n$ for all $x' = (x_1, \ldots, x_n, t) \in \overline{\Omega}_\tau$, and satisfies (3.20) with a right-hand side in $C^p(\overline{\Omega}_\tau; L^\infty(Q))$. Thus we deduce that $\frac{\partial g_k^A}{\partial x_l'} \in C^p(\overline{\Omega}_\tau; \mathcal{H}^n)$ and obtain (3.19). □

Remark 3.1. Since the functions g_k^A and g_k^B satisfy the equations of the same type, (3.14) and (3.15), respectively, the results of Lemma 3.1 are equally valid for the function g_k^B. Likewise, the solutions of the problem (3.11)–(3.13), $\mathcal{N}_k^A$ and $\mathcal{N}_k^B$, also exist and are as regular as the solutions of (3.14) and (3.15).

3.2. *Derivation of homogenized equation*

The third term of ansatz (3.1), $u^{(2)}(x,t,\xi)$, solves the equation which results from equating the terms of order ε^0 after substituting (3.1) into (2.1). This equation is of the form

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial \dot{u}^{(2)}}{\partial \xi_j}\right) - \frac{\partial}{\partial \xi_i}\left(A_{ij}(x,t,\xi)\frac{\partial u^{(2)}}{\partial \xi_j}\right) = \mathcal{F}(x,t,\xi). \tag{3.22}$$

Here

$$\begin{aligned}\mathcal{F}(x,t,\xi) = {} & \frac{\partial}{\partial x_i}\left(B_{ij}(x,t,\xi)\left(\frac{\partial \dot{v}}{\partial x_j} + \frac{\partial \dot{N}}{\partial \xi_j}\right)\right) + \frac{\partial}{\partial x_i}\left(A_{ij}(x,t,\xi)\left(\frac{\partial v}{\partial x_j} + \frac{\partial N}{\partial \xi_j}\right)\right) \\ & + \frac{\partial}{\partial \xi_i}\left(A_{ij}(x,t,\xi)\frac{\partial N}{\partial x_j}\right) + \frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial \dot{N}}{\partial x_j}\right) + f - \rho\ddot{v}. \end{aligned} \tag{3.23}$$

The homogenized equation for v is obtained following a standard recipe as a necessary condition for the existence of $u^{(2)}(x,t,\xi)$ as a solution of problem (3.22) and (3.23) with the initial condition

$$u^{(2)}(x,0,\xi) = 0. \tag{3.24}$$

Indeed, cf. Theorem 3.1, in order for $u^{(2)}(x,t,\xi)$ satisfying (3.22) to exist, the function $\mathcal{F}(x,t,\xi)$ should have zero mean value with respect to ξ over Q:

$$\langle \mathcal{F}(x,t,\xi)\rangle_\xi = 0. \tag{3.25}$$

Substituting (3.23) into (3.25) and using (3.10) and (3.16), we obtain:

$$\widehat{\rho}\,\ddot{v}\,(x,t) - \frac{\partial}{\partial x_i}\sigma_i(x,t) = f(x,t), \tag{3.26}$$

where $\widehat{\rho} = \langle \rho\rangle_\xi$ and

$$\begin{aligned}\sigma_i(x,t) \;=\; & \widehat{B}_{ij}(x,t)\frac{\partial \dot{v}}{\partial x_j}(x,t) + \hat{\mathrm{A}}_{ij}(x,t)\frac{\partial v}{\partial x_j}(x,t)\\ & + \int_0^t \left(\hat{\mathrm{E}}_{ij}(x,t,t')\frac{\partial \dot{v}}{\partial x_j}(x,t') + \widehat{D}_{ij}(x,t,t')\frac{\partial v}{\partial x_j}(x,t')\right)dt'.\end{aligned} \tag{3.27}$$

Importantly, the homogenized relations (3.26) and (3.27) display the "memory effect" due to the integral terms in (3.27).

In (3.27) the following notation has been adopted

$$\begin{aligned}\hat{\mathrm{A}}_{ij}(x,t) \;&:=\; \left\langle A_{ij}(x,t,\xi) + B_{ik}(x,t,\xi)\frac{\partial g_j^A}{\partial \xi_k}\right\rangle_\xi,\\ \widehat{B}_{ij} \;&:=\; \left\langle B_{ik}(x,t,\xi)\left(\delta_{kj}I + \frac{\partial g_j^B}{\partial \xi_k}\right)\right\rangle_\xi\end{aligned} \tag{3.27a}$$

(here $\widehat{B}_{ij}$ is the "conventional" homogenized tensor for B_{ij}, cf. Ref. 5), and

$$\begin{aligned}\hat{\mathrm{E}}_{ij}(x,t,t') \;:=\; & \left\langle A_{ik}(x,t,\xi)\frac{\partial \mathcal{N}_j^B}{\partial \xi_k}(x,t-t',t',\xi)\right\rangle_\xi\\ & + \left\langle B_{ik}(x,t,\xi)\frac{\partial^2 \mathcal{N}_j^B}{\partial \xi_k \partial t}(x,t-t',t',\xi)\right\rangle_\xi,\\ \widehat{D}_{ij}(x,t,t') \;:=\; & \left\langle A_{ik}(x,t,\xi)\frac{\partial \mathcal{N}_j^A}{\partial \xi_k}(x,t-t',t',\xi)\right\rangle_\xi\\ & + \left\langle B_{ik}(x,t,\xi)\frac{\partial^2 \mathcal{N}_j^A}{\partial \xi_k \partial t}(x,t-t',t',\xi)\right\rangle_\xi.\end{aligned} \tag{3.27b}$$

The above memory terms are consistent with those derived for a particular case of scalar problems with time-independent coefficients in e.g. Ref. 13, which are known to be generally present, see Ref. 13, Sec. 4 for some explicit examples.

3.3. *Existence and uniqueness of the homogenized solution*

In this section, we study the existence and uniqueness of the homogenized solution v, which satisfies the following problem formally derived above:

$$\widehat{\rho}\,\ddot{v}(x,t) - \frac{\partial \sigma_i}{\partial x_i}(x,t) = f(x,t), \quad v_{|_{\partial\Omega\times(0,\tau)}} = 0, \quad v_{|_{t=0}} = \dot{v}_{|_{t=0}} = 0, \tag{3.28}$$

together with the constitutive relations (3.27) with "memory".

Theorem 3.2. *Let $f \in L^2(0,\tau;\mathbf{H}^{-1}(\Omega))$ such that there exists a constant $\tau^* < \tau$, such that $f(x,t) = 0$ for all $t \le \tau^*$, and assume that* (H1)–(H3) *hold. Then problem* (3.28) *admits a unique solution $v \in H^1(0,\tau;\mathbf{H}_0^1(\Omega))$.*

Proof. For any v in $H^1(0,\tau;\mathbf{H}_0^1(\Omega))$, we define the vectorial function h^v by

$$h_i^v(x,t) := \int_0^t \left(\hat{\mathrm{E}}_{ij}(x,t,t')\frac{\partial \dot{v}}{\partial x_j}(x,t') + \widehat{D}_{ij}(x,t,t')\frac{\partial v}{\partial x_j}(x,t') \right) dt', \quad i = 1,\ldots,n,$$

and introduce the linear mapping $\mathcal{L} : H^1(0,\tau;\mathbf{H}_0^1(\Omega)) \to H^1(0,\tau;\mathbf{H}_0^1(\Omega))$ by the relation $\mathcal{L}(v) = \tilde{v}$, where $\tilde{v}$ satisfies the following problem:

$$\widehat{\rho}\,\ddot{\tilde{v}} - \frac{\partial}{\partial x_i}\left(\widehat{B}_{ij}(x,t)\frac{\partial \dot{\tilde{v}}}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\hat{\mathrm{A}}_{ij}(x,t)\frac{\partial \tilde{v}}{\partial x_j}\right) = f(x,t) + \frac{\partial h_i^v}{\partial x_i}(x,t), \tag{3.29}$$

$$\tilde{v} = 0 \quad \text{on } \partial\Omega, \tag{3.30}$$

$$\tilde{v}|_{t=0} = 0, \quad \dot{\tilde{v}}|_{t=0} = 0. \tag{3.31}$$

We start the proof by showing that the homogenized coefficients $\widehat{\rho}$, $\hat{\mathrm{A}}_{ij}$, and $\widehat{B}_{ij}$ satisfy the same conditions as those imposed on the coefficients ρ, A_{ij}, and B_{ij} of the original problem. It is easily checked that $\widehat{\rho}$ satisfies assumption (H3).

For the symmetry of $\widehat{B}_{ij}$, first we use (3.27a) to verify that $\widehat{B}_{ij}^{ms} = \widehat{B}_{mj}^{is}$ for all $1 \le i,j,m,s \le n$. Second, we use the weak formulation of (3.15) (see (3.20) and Ref. 24, p. 151) to justify that $\forall\, 1 \le i,j,m,s \le n$

$$\widehat{B}_{ij}^{ms} = \langle B_{lk}^{qp} \mathcal{Z}_k^{ps}(g_j^B + \xi_j I)\mathcal{Z}_l^{qm}(g_i^B + \xi_i I)\rangle_\xi, \tag{3.32}$$

where

$$\mathcal{Z}_k^{ps}(g_j + \xi_j I) := \frac{1}{2}\left[\frac{\partial}{\partial \xi_k}(g_j^{ps} + \xi_j\delta_{ps}) + \frac{\partial}{\partial \xi_p}(g_j^{ks} + \xi_j\delta_{ks})\right].$$

Making use of the relation $B_{lk}^{qp} = B_{kl}^{pq}$ in (3.32), we get $\widehat{B}_{ij}^{ms} = \widehat{B}_{ji}^{sm}$.

Let $\eta = (\eta_i^m) \in \mathbb{R}^{n\times n}$, such that $\eta_i^m = \eta_m^i$. Taking (H2) into account, we obtain from (3.32) the following inequality:

$$\nu\eta_i^m\eta_i^m \le \widehat{B}_{ij}^{ms}\eta_i^m\eta_j^s \le 2\nu^{-1}\left(1 + n\max_{1\le i\le n}\|g_i^B\|_{\mathcal{H}^n}^2\right)\eta_i^m\eta_i^m. \tag{3.33}$$

Thus, by using (3.21) in the last inequality, we prove that $\widehat{B}_{ij}$ are uniformly bounded and uniformly elliptic. (Notice in passing that the above properties of symmetry, uniform ellipticity and boundedness for the homogenized tensor are known to hold for "classical" elliptic problems of linear elasticity, e.g. Ref. 5, according to which recipe $\hat{B}$ is associated with B, as mentioned above.) Similarly, it suffices to use (3.18) in the definitions of $\hat{\mathrm{A}}_{ij}$ and the time derivatives of $\hat{\mathrm{A}}_{ij}$ and $\widehat{B}_{ij}$ to prove that they are uniformly bounded on $\mathcal{H}^n$ by a constant c depending only on ν. Thus we have proved that, under the assumptions of the theorem, the homogenized coefficients satisfy assumptions (H1)−(H3) (although the constants in the upper bounds may not be the same).

Now we can apply Theorem 1.1 to problem (3.29)−(3.31) for $\tilde{v}$ with a given v. In particular, if $f \equiv 0$, then

$$\|\tilde{v}\|_{H^1(0,\tau;\mathbf{H}_0^1(\Omega))} \le C_1\|h^v\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))}.$$

By (3.18) the functions $\mathcal{N}_k^A, \mathcal{N}_k^B$ and their time derivatives $\dot{\mathcal{N}}_k^A, \dot{\mathcal{N}}_k^B$ are uniformly bounded, and so the latter inequality implies that there exists a constant C_3 depending only on ν, ρ_1, τ, and Ω (this constant C_3 tends to zero when $\tau \to 0$), such that

$$\|\tilde{v}\|_{H^1(0,\tau;\mathbf{H}_0^1(\Omega))} \le C_3\|v\|_{H^1(0,\tau;\mathbf{H}_0^1(\Omega))}. \tag{3.34}$$

Let $\tau = \tau_1 > 0$ be small enough for C_3 to satisfy $C_3 < 1$, implying that for any given f the mapping $\mathcal{L}$ is a contraction. Then the Banach fixed point theorem implies that $\mathcal{L}$ has a unique fixed point $v = \tilde{v}$, which is the solution of problem (3.28) on Ω_{τ_1}. Since for $t \in [0, \tau]$, we have $v(t) \in \mathbf{H}_0^1(\Omega)$, we can then repeat the above argument to extend our solution to the time interval $[\tau_1, 2\tau_1]$, and so on. After a finite number of steps we construct a solution existing on the interval $[0, \tau]$. To prove uniqueness, it is enough to take into account that $\tilde{v} = v$ in (3.34) and deduce that for $C_3 < 1$, the zero function is the unique solution of problem (3.28) with $f \equiv 0$. □

3.4. *Justification of the asymptotics*

If v solves the homogenized problem (3.28), then, by its derivation, the solvability condition for Eq. (3.22) for $u^{(2)}$ is satisfied. Hence there exists a solution $u^{(2)}(x, t, \xi)$ of (3.22).

Consider the following representation for the exact solution:

$$u^\varepsilon(x,t) = v(x,t) + \varepsilon N\left(x, t, \frac{x}{\varepsilon}\right) + \varepsilon^2 u^{(2)}\left(x, t, \frac{x}{\varepsilon}\right) + r^\varepsilon(x,t). \tag{3.35}$$

Our aim is to obtain an estimate for the "remainder" $r^\varepsilon(x,t)$ for ε sufficiently small.

In order to justify the asymptotic expansion of u^ε we will impose additional regularity assumptions on the viscoelastic coefficients. Namely, we will assume that

A_{ij} and B_{ij} are both smooth with respect to x and t, and periodic and piecewise smooth with respect to ξ. More precisely, we will assume that there exist disjoint periodic subdomains $\mathcal{D}_m \subset \mathbb{R}^n, m = 1,\dots,L$, such that $\mathbb{R}^n = \cup_{m=1}^{L} \overline{\mathcal{D}}_m$ and that each $\mathcal{D}_m$ is in Hölder class $C^{1,\beta}(\overline{\mathcal{D}}_m)$ of periodic functions, with $0 < \beta \le 1$. We hence require the physical characteristics of the composite media to be smooth (e.g. constant) in ξ in each subdomain $\overline{\mathcal{D}}_m$ assumed itself having a sufficiently smooth boundary, but possibly discontinuous across their boundaries.

Now by first assuming that $v(x,t), N(x,t,\xi)$ and $u^{(2)}(x,t,\xi)$ are smooth with respect to x and t in Ω_τ and piecewise smooth with respect to ξ, we will prove the convergence of u^ε to v when the parameter ε tends to zero and establish the relevant error bounds. In the next section we will describe sufficient conditions which ensure the validity of these assumptions on $v(x,t), N(x,t,\xi)$ and $u^{(2)}(x,t,\xi)$.

Theorem 3.3. *We assume that*:

(H4) v, $\dot v \in C^3(\overline{\Omega}_\tau); N, \dot N \in C^2(\overline{\Omega}_\tau;\mathcal{K}); u^{(2)}, \dot u^{(2)} \in C^1(\overline{\Omega}_\tau;\mathcal{K})$; *and for* $m = 1,\dots,$ L, *we have* $A_{ij}, B_{ij} \in C^2(\overline{\Omega}_\tau; C^{1,\lambda}(\overline{\mathcal{D}}_m))$, *with* $0 < \lambda < 1$.

Then there exists a constant C independent of ε such that

$$\|u^\varepsilon - (v + \varepsilon N)\|_{L^\infty(0,\tau;\mathbf{H}_0^1(\Omega))} + \|\dot u^\varepsilon - (\dot v + \varepsilon \dot N)\|_{L^2(0,\tau;\mathbf{H}_0^1(\Omega))} \le C\varepsilon^{1/2},$$

$$\|u^\varepsilon - v\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} + \|\dot u^\varepsilon - \dot v\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} \le C\varepsilon^{1/2}.$$

Henceforth, we denote for a fixed $0 < \zeta < 1$

$$\mathcal{K} := \{u \in C^{1,\zeta}(\overline{\mathcal{D}}_m), m = 1,\dots,L\}.$$

Proof. Let $\chi_\varepsilon(x)$ be a differentiable function whose support belongs to the ε-neighborhood of the boundary of Ω, such that $\chi_\varepsilon|_{\partial\Omega} = 1$, $|\chi_\varepsilon| \le 1$, and $\|\varepsilon \frac{\partial \chi_\varepsilon}{\partial x_j}\|_{C(\Omega)} \le c$, with a constant c independent of ε. (Such a "cut-off function" χ_ε exists, see e.g. Refs. 5 and 29.) Set

$$\tilde u_2^\varepsilon(x,t) = v(x,t) + (1 - \chi_\varepsilon(x))\Big(\varepsilon N\Big(x,t,\frac{x}{\varepsilon}\Big) + \varepsilon^2 u^{(2)}\Big(x,t,\frac{x}{\varepsilon}\Big)\Big),$$

noticing that $\tilde u_2^\varepsilon$ thereby satisfies the boundary condition (2.2).

By using (H4) we notice that for all $m = 1,\dots,L$ we have: A_{ij}, B_{ij}, N, $\dot N, u^{(2)}$, and $u^{(2)}$ belong to $C^1(\overline{\Omega}_\tau \times \overline{\mathcal{D}}_m)$, hence one can use the chain rule: $\frac{\partial}{\partial x_i} w(x, x/\varepsilon) = [\frac{\partial}{\partial x_i} w(x,\xi)]_{\xi=x/\varepsilon} + \varepsilon^{-1}[\frac{\partial}{\partial \xi_i} w(x,\xi)]_{\xi=x/\varepsilon}$, for any of the above functions. Thus, we substitute $\tilde r^\varepsilon := u^\varepsilon - \tilde u_2^\varepsilon$ into the original equation (2.1).

According to the derivation of the terms of expansion (3.35), the terms of the order ε^{-1} and ε^0 will vanish. As a result, taking into account (2.1)–(2.3), (3.28), and the zero initial conditions for N and $u^{(2)}$, we obtain the following problem for $\tilde r^\varepsilon(x,t)$:

$$\rho \ddot{\tilde r}^\varepsilon - \frac{\partial}{\partial x_i}\left(B_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial \dot{\tilde r}^\varepsilon}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(A_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial \tilde r^\varepsilon}{\partial x_j}\right) = h_\varepsilon^1 + \frac{\partial h_\varepsilon^2}{\partial x_i} + \frac{\partial h_\varepsilon^3}{\partial x_i}, \quad (3.36)$$

$$\tilde{r}^{\varepsilon} = 0 \quad \text{on } \partial\Omega, \quad \tilde{r}^{\varepsilon}(x,0) = 0, \tag{3.37}$$

$$\dot{\tilde{r}}^{\varepsilon}(x,0) = \varepsilon(\chi_{\varepsilon}(x) - 1)\dot{N}\Big(x, 0, \frac{x}{\varepsilon}\Big) + \varepsilon^2(\chi_{\varepsilon} - 1)\dot{u}^{(2)}\Big(x, 0, \frac{x}{\varepsilon}\Big) \quad \text{on } \Omega. \tag{3.38}$$

Here,

$$\begin{aligned} h^1_{\varepsilon}(x,t) \,:=\, & \varepsilon\bigg[\frac{\partial}{\partial x_i}\bigg(A_{ij}(x,t,\xi)\frac{\partial N}{\partial x_j}(x,t,\xi) + B_{ij}(x,t,\xi)\frac{\partial \dot{N}}{\partial x_j}(x,t,\xi)\bigg)\bigg]_{\xi=\frac{x}{\varepsilon}} \\ & + \varepsilon\bigg[\frac{\partial}{\partial x_i}\bigg(A_{ij}(x,t,\xi)\frac{\partial u^{(2)}}{\partial \xi_j}(x,t,\xi) + B_{ij}(x,t,\xi)\frac{\partial \dot{u}^{(2)}}{\partial \xi_j}(x,t,\xi)\bigg)\bigg]_{\xi=\frac{x}{\varepsilon}} \\ & + \rho\Big(\frac{x}{\varepsilon}\Big)(\chi_{\varepsilon}(x) - 1)\Big(\varepsilon\ddot{N}\Big(x,t,\frac{x}{\varepsilon}\Big) + \varepsilon^2\ddot{u}^{(2)}\Big(x,t,\frac{x}{\varepsilon}\Big)\Big), \end{aligned}$$

$$h^2_{\varepsilon}(x,t) := \varepsilon^2\bigg[A_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial u^{(2)}}{\partial x_j}(x,t,\xi) + B_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial \dot{u}^{(2)}}{\partial x_j}(x,t,\xi)\bigg]_{\xi=\frac{x}{\varepsilon}},$$

and

$$\begin{aligned} h^3_{\varepsilon}(x,t) \,:=\, & -A_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}\Big[\chi_{\varepsilon}(x)(\varepsilon N + \varepsilon^2 u^{(2)})\Big(x,t,\frac{x}{\varepsilon}\Big)\Big] \\ & - B_{ij}\Big(x,t,\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}\Big[\chi_{\varepsilon}(x)(\varepsilon \dot{N} + \varepsilon^2 \dot{u}^{(2)})\Big(x,t,\frac{x}{\varepsilon}\Big)\Big]. \end{aligned}$$

Now, using (H4), we establish the following estimate:

$$\|h^1_{\varepsilon}\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))} + \|h^2_{\varepsilon}\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))} \le c\varepsilon, \tag{3.39}$$

with c denoting, henceforth, a constant whose precise value is insignificant and may change from line to line.

Further, taking into account the above described properties of χ_{ε}, we obtain (cf. e.g. Refs. 5 and 29)

$$\|h^3_{\varepsilon}\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))} \le c\varepsilon^{1/2}, \quad \|\tilde{r}^{\varepsilon}(0)\|_{\mathbf{H}^1_0(\Omega)} + \|\dot{\tilde{r}}^{\varepsilon}(0)\|_{\mathbf{L}^2(\Omega)} \le c\varepsilon. \tag{3.40}$$

Thus, applying Theorem 2.1 to problem (3.36)–(3.38) and using estimates (3.39) and (3.40), we obtain the following inequality:

$$\|\tilde{r}^{\varepsilon}\|_{L^\infty(0,\tau;\mathbf{H}^1_0(\Omega))} + \|\dot{\tilde{r}}^{\varepsilon}\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} + \|\dot{\tilde{r}}^{\varepsilon}\|_{L^2(0,\tau;\mathbf{H}^1_0(\Omega))} \le c\varepsilon^{1/2}.$$

Finally, we take into account that

$$\|\tilde{u}^{\varepsilon}_2 - (v + \varepsilon N)\|_{L^\infty(0,\tau;\mathbf{H}^1_0(\Omega))} + \|\dot{\tilde{u}}^{\varepsilon}_2 - (\dot{v} + \varepsilon \dot{N})\|_{L^2(0,\tau;\mathbf{H}^1_0(\Omega))} \le c\varepsilon^{1/2}$$

and

$$\|\tilde{u}^{\varepsilon}_2 - v\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} + \|\dot{\tilde{u}}^{\varepsilon}_2 - \dot{v}\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} \le c\varepsilon$$

to arrive at the claimed estimates. □

4. Sufficient Condition for Regularity

In this section, we study the regularity properties of the terms of asymptotic expansion (3.1). In particular, we are interested in sufficient conditions for v to satisfy the assumptions of Theorem 3.3 above. The appearance of a long memory "integral term" in the homogenized equations (3.26) and (3.27) presents additional technical complications for the study of the regularity of v, N, and $u^{(2)}$. To simplify the matters, we prove here that the required regularity holds at least in a particular case. We expect that, essentially, the following argument can be adjusted to the general case, at the expense of conceptually rather straightforward although technically involved modifications. Namely, we consider the case when the elastic and viscous characteristics are proportional, i.e. there exists a constant $\kappa > 0$ such that

$$A_{ij} = \kappa B_{ij}, \quad i, j = 1, \ldots, n. \tag{4.1}$$

In this case (2.1) takes the form

$$\rho\Big(\frac{x}{\varepsilon}\Big)\ddot{u}^\varepsilon - \frac{\partial}{\partial x_i}\left(B_{ij}\Big(x, t, \frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}(\dot{u}^\varepsilon + \kappa u^\varepsilon)\right) = f(x, t).$$

Moreover, the main corrector of the asymptotic expansion (3.1) is in this case given by

$$N(x, t, \xi) = \int_0^t e^{-\kappa(t-t')} g_k(x, t', \xi)\frac{\partial}{\partial x_k}(\dot{v} + \kappa v)(x, t')\, dt',$$

where, for any x in Ω, the function $g_k(x, \cdot, \cdot) \in H^1(0, \tau; \mathcal{H}^n)$ satisfies the equation

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x, t, \xi)\frac{\partial g_k}{\partial \xi_j}\right) = \frac{\partial}{\partial \xi_i} B_{ik}(x, t, \xi). \tag{4.2}$$

The latter can be verified by direct inspection of (3.7)–(3.9) when (4.1) is held.

Combined with expressions (3.27a) and (3.27b) for the homogenized coefficients $\hat{\mathrm{A}}_{ij}$, $\widehat{B}_{ij}$, $\widehat{D}_{ij}$, and $\hat{\mathrm{E}}_{ij}$, the latter formulae imply via (3.11)–(3.15):

$$\widehat{B}_{ij} = \kappa^{-1}\hat{\mathrm{A}}_{ij} = \left\langle B_{ij} + B_{ik}\frac{\partial g_j}{\partial \xi_k}\right\rangle_\xi, \quad \hat{\mathrm{E}}_{ij} = \widehat{D}_{ij} = 0.$$

Substituting these into (3.26) and (3.27), we conclude that the homogenized equation takes the form

$$\widehat{\rho}\,\ddot{v}(x, t) - \frac{\partial}{\partial x_i}\left(\widehat{B}_{ij}(x, t)\frac{\partial}{\partial x_j}(\dot{v} + \kappa v)\right) = f(x, t), \tag{4.3}$$

with the memory term vanishing. (Remind in passing that the memory term does *not* generally vanish, see Ref. 13, Sec. 4.)

Lemma 4.1. *Let $m \in \mathbb{N}^*$ such that $m > \frac{n}{2}$, and let Ω be an open bounded set of $\mathbb{R}^n$ with C^{2m+2} boundary. Assume that $B_{ij} \in C^{m+1}(\overline{\Omega}_\tau; L^\infty(Q))$ and $f^{(k)} \in L^2(0, \tau;$*

$\mathbf{H}^{2m-2k}(\Omega)), k = 0, \ldots, m$ (*here* $f^{(k)}$ *is the* k-*th time derivative of* f), *such that there exists a constant* $\tau^* < \tau$, *such that* $f(x,t) = 0$ *for all* $t \leq \tau^*$. *Then*

$$v, \dot{v} \in C^{m-[\frac{n}{2}]}(\overline{\Omega}_\tau).$$

Proof. Let v be a solution of (4.3) supplemented by boundary and initial conditions as in (3.28). By Theorem 3.2, v exists and is unique in $H^1(0,\tau;\mathbf{H}_0^1(\Omega))$. Throughout the proof we will use the following notation: $\tilde{v} := \dot{v} + \kappa\, v$ and $\tilde{f} := f + \kappa\,\widehat{\rho}\dot{v}$. Thus $\tilde{v}$ satisfies

$$\widehat{\rho}\dot{\tilde{v}} - \frac{\partial}{\partial x_i}\left(\widehat{B}_{ij}\frac{\partial \tilde{v}}{\partial x_j}\right) = \tilde{f}, \quad \tilde{v} = 0, \quad x \in \partial\Omega, \quad \tilde{v}_{|_{t=0}} = 0. \tag{4.4}$$

The lemma is known to hold in the case when the coefficients $\widehat{B}_{ij}$ are independent of time (see e.g. Ref. 14).

We will use the Galerkin's method. Let $B_{ij} \in C^{m+1}(\Omega_\tau)$ and $\tilde{f}^{(k)} \in L^2(0,\tau;\mathbf{H}^{2m-2k}(\Omega)), k = 0, \ldots, m$. We will then show that

$$\tilde{v}^{(k)} \in L^2(0,\tau;\mathbf{H}^{2m+2-2k}(\Omega)), \quad k = 0, \ldots, m+1, \tag{4.5}$$

and there exists a constant C depending only on m, τ, Ω, and ν, such that

$$\sum_{k=0}^{m+1}\|\tilde{v}^{(k)}\|_{L^2(0,\tau;\mathbf{H}^{2m+2-2k}(\Omega))} \leq C\sum_{k=0}^{m}\|\tilde{f}^{(k)}\|_{L^2(0,\tau;\mathbf{H}^{2m-2k}(\Omega))}. \tag{4.6}$$

First, notice that Lemma 3.3 implies that if $B_{ij} \in C^{m+1}(\overline{\Omega}_\tau; Q)$ then the function g_k solving (4.2) belongs to $C^{m+1}(\overline{\Omega}_\tau)$, thus we have $\widehat{B}_{ij} \in C^{m+1}(\overline{\Omega}_\tau)$.

(i) Let $f \in L^2(0,\tau;\mathbf{L}^2(\Omega))$ and $B_{ij} \in C^1(\overline{\Omega}_\tau)$, and let us prove that (4.5) and (4.6) hold for $m = 0$. Let $\tilde{v} \in L^2(0,\tau;\mathbf{H}_0^1(\Omega)) \cap C([0,\tau];\mathbf{L}^2(\Omega))$ be a weak solution of (4.4), i.e.

$$(\widehat{\rho}\dot{\tilde{v}}, z)_\Omega + \mathcal{I}_{\widehat{B}}(\tilde{v}, z)_\Omega = (\tilde{f}, z)_\Omega, \ \ \forall\, z \in \mathbf{H}_0^1(\Omega) \quad \text{in } \mathcal{D}'(0,\tau) \text{ and } \tilde{v}(0) = 0. \tag{4.7}$$

Hence $\tilde{v} \in H^1(0,\tau;\mathbf{H}_0^1(\Omega))$ and also $\tilde{f} \in L^2(0,\tau;\mathbf{L}^2(\Omega))$.

Thus, by using (H1), we prove that for all integer $p \geq 1$ there exists a unique function:

$$\tilde{v}_p(t) = \sum_{i=1}^{p} d_p^i(t) w_i(x) \in H^1(0,\tau;\,\mathbf{H}_0^1(\Omega)),$$

solving the approximate Galerkin's problem: for $j = 1, \ldots, p$,

$$(\widehat{\rho}\,\dot{\tilde{v}}_p(t), w_j)_\Omega + \mathcal{I}_{\widehat{B}}(\tilde{v}_p(t), w_j)_\Omega = (\tilde{f}(t), w_j)_\Omega \quad \text{in } \mathcal{D}'(0,\tau) \text{ and } \tilde{v}_p(0) = 0. \tag{4.8}$$

Here $(w_i)_{i\geq 0}$ is an orthogonal basis of $H_0^1(\Omega)$.

Multiplying (4.8) by $\dot{d}_p^j$ and summing over $j = 1, \ldots, p$, we obtain

$$(\widehat{\rho}\dot{\tilde{v}}_p, \dot{\tilde{v}}_p)_\Omega + \mathcal{I}_{\widehat{B}}(\tilde{v}_p, \dot{\tilde{v}}_p)_\Omega = (\tilde{f}, \dot{\tilde{v}}_p)_\Omega.$$

Then, by using (2.10) and Gronwall's lemma (cf. proof of Theorem 2.1), we deduce that there exists a constant c which depends only on ν, τ, ρ_1, and Ω, such that

$$\|\tilde{v}_p\|_{L^\infty(0,\tau;\mathbf{H}_0^1(\Omega))} + \|\dot{\tilde{v}}_p\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))} \leq c\|\tilde{f}\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))}. \tag{4.9}$$

The last estimate allows us to pass to the limit as $p \to \infty$ in a standard way and to check that $\tilde{v} \in L^\infty(0,\tau;\mathbf{H}_0^1(\Omega)), \dot{\tilde{v}} \in L^2(0,\tau;\mathbf{L}^2(\Omega))$ and satisfy (4.7). Thus, for almost all $t \in [0,\tau]$ we have $\tilde{f}(t) - \widehat{\rho}\dot{\tilde{v}}(t) \in \mathbf{L}^2(\Omega)$ and according to an elliptic regularity result (Ref. 14, Sec. 6.3.2, Theorem 4), we obtain from (4.7)

$$\|\tilde{v}\|_{\mathbf{H}^2(\Omega)} \leq c(\|\tilde{f}\|_{\mathbf{L}^2(\Omega)} + \|\dot{\tilde{v}}\|_{\mathbf{L}^2(\Omega)} + \|\tilde{v}\|_{\mathbf{L}^2(\Omega)}).$$

Finally, we integrate the last inequality over $[0,\tau]$ and employ (4.9) to obtain

$$\|\tilde{v}\|_{L^2(0,\tau;\mathbf{H}^2(\Omega))} + \|\dot{\tilde{v}}\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))} \leq C\|\tilde{f}\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))},$$

which completes the proof of (4.5) and (4.6) for $m = 0$.

(ii) Assume that (4.5) and (4.6) are valid for $m \geq 0$ and let $B_{ij} \in C^{m+2}(\overline{\Omega}_\tau)$, $\tilde{f}^{(k)} \in L^2(0,\tau;\mathbf{H}^{2m+2-2k}(\Omega)), k = 0,\ldots,m+1$. Differentiate (4.7) with respect to t and then check that $\dot{\tilde{v}}$ satisfies (4.7) with the right-hand side defined by:

$$\tilde{F} = \dot{f} + \kappa\widehat{\rho}\,\ddot{v} + \frac{\partial}{\partial x_i}\left(\dot{\widehat{B}}_{ij}\frac{\partial \tilde{v}}{\partial x_j}\right).$$

Relations (4.5) and (4.6) of order m imply that for $k = 0,\ldots,m+1$, the function $\tilde{v}^{(k)}$ belongs to $L^2(0,\tau;\mathbf{H}^{2m+2-2k}(\Omega))$. Then, we deduce that $\tilde{F}^{(k)}$ belongs to L^2 $(0,\tau;\mathbf{H}^{2m-2k}(\Omega)), k = 0,\ldots,m$. Further, from (4.4) at $t = 0$, we deduce that $\dot{\tilde{v}}(0) = 0$.

Thus by applying the induction assumption we get

$$\tilde{v}^{(k)} \in L^2(0,\tau;\mathbf{H}^{2m+4-2k}(\Omega)), \quad k = 1,\ldots,m+2.$$

Taking into account (4.6), we obtain

$$\sum_{k=1}^{m+2}\|\tilde{v}^{(k)}\|_{L^2(0,\tau;\mathbf{H}^{2m+4-2k}(\Omega))} \leq C\sum_{k=0}^{m+1}\|\tilde{f}^{(k)}\|_{L^2(0,\tau;\mathbf{H}^{2m+2-2k}(\Omega))}. \tag{4.10}$$

According to (Ref. 14, Sec. 6.3.2), we have

$$\|\tilde{v}\|_{\mathbf{H}^{2m+4}(\Omega)} \leq c(\|\tilde{f}\|_{\mathbf{H}^{2m+2}(\Omega)} + \|\dot{\tilde{v}}\|_{\mathbf{H}^{2m+2}(\Omega)} + \|\tilde{v}\|^2_{\mathbf{L}^2(\Omega)}).$$

Finally, we integrate the last inequality over $[0,\tau]$ and use (4.10) to assert the following estimate:

$$\sum_{k=0}^{m+2}\|\tilde{v}^{(k)}\|_{L^2(0,\tau;\mathbf{H}^{2m+4-2k}(\Omega))} \leq C\sum_{k=0}^{m+1}\|\tilde{f}^{(k)}\|_{L^2(0,\tau;\mathbf{H}^{2m+2-2k}(\Omega))}.$$

Thus (4.5) and (4.6) are proved. Finally, we use these results to prove the lemma. Indeed, since Eq. (4.3) is accompanied by trivial initial conditions ($v(0) = \dot{v}(0) = 0$), we have $\tilde{v}(0) = 0$. It results from (4.5) and (4.6) that under the assumptions of the lemma we have

$$v, \dot{v} \in H^{m+1}(0, \tau; \mathbf{H}^{2m+2}(\Omega)),$$

and therefore we deduce the assertion of the lemma by the Sobolev embedding. □

Remark 4.1. Lemma 4.1 gives the following condition for the inclusions v, $\dot{v} \in C^3(\overline{\Omega}_\tau)$ which are assumed in (H4): relations (4.1) and inclusions $A_{ij}, B_{ij} \in C^{[\frac{n}{2}]+4}(\overline{\Omega}_\tau; L^\infty(Q))$, and for all $k = 0, 1, 2, 3, f^{(k)} \in L^2(0, \tau; \mathbf{H}^{6-2k}(\Omega))$. Henceforth, we assume that these inclusions are valid.

Lemma 4.2. *Let $\mathcal{D}_m \subset \mathbb{R}^n, m = 1, \ldots, L$ be disjoint periodic subdomains such that each $\mathcal{D}_m$ is of class $C^{1,\beta}$ with $0 < \beta \leq 1$ and $\mathbb{R}^n = \cup_{m=1}^L \overline{\mathcal{D}}_m$. We also assume that $B_{ij}(x, t, \xi)$ are one-periodic with respect to ξ, such that $B_{ij} \in C^2(\overline{\Omega}_\tau; C^{0,\lambda}(\overline{\mathcal{D}}_m)), 0 < \lambda < 1, m = 1, \ldots, L$. Then*

$$N, \dot{N} \in C^2(\overline{\Omega}_\tau; \mathcal{K}), \quad u^{(2)}, \dot{u}^{(2)} \in C^1(\overline{\Omega}_\tau; \mathcal{K}).$$

Proof. By taking account of (4.1) in (3.7) we check that for all $(x, t) \in \overline{\Omega}_\tau$ the function $N \in \mathbf{H}^1_\#(Q)$ satisfies the following equation

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x, t, \xi)\frac{\partial}{\partial \xi_j}\left(\dot{N} + \kappa N\right)\right) = \frac{\partial}{\partial \xi_i}\left(B_{ij}(x, t, \xi)\frac{\partial}{\partial x_j}(\dot{v} + \kappa v)\right). \tag{4.11}$$

Let $\mathcal{D} \subset \mathbb{R}^n$ be a bounded subdomain such that $\overline{Q} \Subset \mathcal{D}$. By the periodicity property of N and B_{ij}, we can study (4.11) in $\mathcal{D}$.

By assumption, for all (x, t) in $\overline{\Omega}_\tau$ we have $B_{ij}(x, t, \cdot) \in C^{0,\lambda}(\overline{\mathcal{D}}_m), m = 1, \ldots, L$. Then according to Ref. 19 (see also Ref. 25) we conclude that for all $(x, t) \in \overline{\Omega}_\tau$, for any $\delta > 0$ and for all $0 < \lambda_1 \leq \min\{\lambda, \frac{\beta}{2(\beta+1)}\}$, multiplying $\dot{N} + \kappa N$ by $e^{-\kappa(t-\tau)}$ and integrating in τ, we have: N, $\dot{N} \in C^{1,\lambda_1}(\overline{\mathcal{D}}_m \cap \mathcal{D}_\delta), m = 1, \ldots, L$, where $\mathcal{D}_\delta = \{\xi \in \mathcal{D};\ \text{dist}(\xi, \partial\mathcal{D}) > \delta\}$. In particular, $\nabla_\xi N$ and $\nabla_\xi \dot{N}$ are bounded.

Let us recall that the earlier assumption $B_{ij} \in C^{[\frac{n}{2}]+4}(\overline{\Omega}_\tau; L^\infty(Q))$ implies that N, $\dot{N} \in C^2(\overline{\Omega}_\tau; \mathbf{H}^1_\#(Q))$ (see Lemma 3.1). Thus we use the continuity of B_{ij} on x and t with respect to the $C^{0,\lambda}(\overline{\mathcal{D}}_m)$-norm and that v, $\dot{v} \in C^1(\overline{\Omega}_\tau)$, and we employ the same techniques as in the proof of Lemma 3.1 to prove that N, $\dot{N} \in C(\overline{\Omega}_\tau; C^{1,\lambda_1}(\overline{\mathcal{D}}_m \cap \mathcal{D}_\delta))$ and that $\nabla_\xi N$, $\nabla_\xi \dot{N} \in C(\overline{\Omega}_\tau; L^\infty(\mathcal{D}_\delta))$.

Similarly, we differentiate separately (4.11) with respect to x and t and we use that $B_{ij} \in C^1(\overline{\Omega}_\tau; C^{0,\lambda}(\overline{\mathcal{D}}_m))$ to check that there exists a constant $0 < \lambda_2 < 1$, such that $\ddot{N}, \dot{N}_x, N_x$ belong to $C(\overline{\Omega}_\tau; C^{1,\lambda_2}(\overline{\mathcal{D}}_m \cap \mathcal{D}_\delta))$ and that $\nabla_\xi \ddot{N}$, $\nabla_\xi \dot{N}_x, \nabla_\xi N_x$ belong to $C(\overline{\Omega}_\tau; L^\infty(\mathcal{D}_\delta))$, having denoted for brevity $\nabla_x N$ by N_x.

In the same way, by differentiating separately (4.11) two times with respect to t and x and using that $B_{ij} \in C^2(\overline{\Omega}_\tau; C^{0,\lambda}(\overline{\mathcal{D}}_m))$ we check that there exists a constant $0 < \lambda_3 < 1$, such that the third time derivative $N^{(3)}$ of N and $N_{xx}, \dot{N}_{xx}$ belong to $C(\overline{\Omega}_\tau; C^{1,\lambda_3}(\overline{\mathcal{D}}_m \cap \mathcal{D}_\delta))$ for $m = 1, \ldots, L$, and that $\nabla_\xi N^{(3)}, \nabla_\xi N_{xx}, \nabla_\xi \dot{N}_{xx}$ belong to $C(\overline{\Omega}_\tau; L^\infty(\mathcal{D}_\delta))$.

Thus we fix δ so that $\overline{Q} \subset \mathcal{D}_\delta$. We deduce from the results above that N, $\dot{N} \in C^2(\overline{\Omega}_\tau; C^{1,\gamma}(\overline{\mathcal{D}}_m))$ for all $m = 1, \ldots, L$, and that $\nabla_\xi N, \nabla_\xi \dot{N} \in C^2(\overline{\Omega}_\tau; L^\infty(\mathcal{D}_\delta))$, (here $\gamma = \min_{1 \le i \le 3} \lambda_i$).

Now we will similarly study the problem of $u^{(2)}$ in $\mathcal{D}_\delta$ that will be denoted by $\mathcal{D}'$. To be done, we use (4.1) to check that (3.22) changes to

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial}{\partial \xi_j}\left(\dot{u}^{(2)} + \kappa u^{(2)}\right)\right) = \mathcal{R}(x,t,\xi) + \frac{\partial \mathcal{S}_i}{\partial \xi_i}(x,t,\xi) \tag{4.12}$$

with

$$\mathcal{R} = \frac{\partial}{\partial x_i}\left(B_{ij}\left(\frac{\partial \dot{v}}{\partial x_j} + \frac{\partial \dot{N}}{\partial \xi_j}\right)\right) + \frac{\partial}{\partial x_i}\left(A_{ij}\left(\frac{\partial v}{\partial x_j} + \frac{\partial N}{\partial \xi_j}\right)\right) + f - \rho \ddot{v}$$

and

$$\mathcal{S}_i = A_{ij}\frac{\partial N}{\partial x_j} + B_{ij}\frac{\partial \dot{N}}{\partial x_j}, \quad i = 1, \ldots, n.$$

We use the assumptions of the lemma and the above results concerning N to check that $\mathcal{R} \in C^1(\overline{\Omega}_\tau; L^\infty(\mathcal{D}'))$ and $\mathcal{S} \in C^1(\overline{\Omega}_\tau; C^{0,\zeta}(\overline{\mathcal{D}}_m)), m = 1, \ldots, p$, with $0 < \zeta < 1$. First, we use Ref. 19 in (4.12) and the continuity of the right-hand side of (4.12) with respect to x and t to conclude that there exists a constant $0 < \lambda_4 < 1$, such that for all $\delta > 0, u^{(2)}, \dot{u}^{(2)} \in C(\overline{\Omega}_\tau; C^{1,\lambda_4}(\overline{\mathcal{D}}_m \cap \mathcal{D}'_\delta)), m = 1, \ldots, p$, and that $\nabla_\xi u^{(2)}, \nabla_\xi \dot{u}^{(2)} \in C(\overline{\Omega}_\tau; L^\infty(\mathcal{D}'_\delta))$. Second, we differentiate separately (4.12) with respect to x and t, and we use the same argument as above to conclude that for all $m = 1, \ldots, p; u_x^{(2)}$, $\dot{u}_x^{(2)}, \ddot{u}^{(2)} \in C(\overline{\Omega}_\tau; C^{1,\lambda_5}(\overline{\mathcal{D}}_m \cap \mathcal{D}'_\delta)), 0 < \lambda_5 < 1$, and that for all $\delta > 0$ we have: $\nabla_\xi u_x^{(2)}, \nabla_\xi \dot{u}_x^{(2)}, \nabla_\xi \ddot{u}^{(2)} \in C(\overline{\Omega}_\tau; L^\infty(\mathcal{D}'_\delta))$.

Finally it is enough to choose a suitable δ so that $\mathcal{D}'_\delta$ contains $\overline{Q}$ and we define $\zeta = \min\{\gamma, \lambda_4, \lambda_5\}$ to obtain the result of the lemma. □

We thereby conclude that under assumptions of Lemmas 4.1 and 4.2, the proportionality condition (4.1) is sufficient to justify the validity of the assumptions made in Theorem 3.3 and consequently it ensures the results of Theorem 3.3. Note finally that in (H4) we require that B_{ij} belong to $C^2(\overline{\Omega}_\tau;\ C^{1,\lambda}(\overline{\mathcal{D}}_m))$, ensuring $B_{ij} \in C^1(\overline{\Omega}_\tau \times \overline{\mathcal{D}}_m)$.

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