MA40189 - Useful results for Markov chains

https://moodle.bath.ac.uk/course/view.php?id=1179

Simon Shaw, s.shaw@bath.ac.uk

2024/25 Semester II

Irreducible

Definition

A Markov chain is said to be **irreducible** *if for every i, j there exists k such that*

$$P(X_{t+k} = j \,|\, X_t = i) \quad > \quad 0$$

that is, all states can be reached from any other state in a finite number of moves.

Periodic/Aperiodic

Definition

A state i is said to be **periodic** *with period d_i if starting from state i the chain returns to it within a fixed number of steps d_i or a multiple of d_i.*

$$d_i \;=\; gcd\{t : P(X_t = i \,|\, X_0 = i) > 0\}$$

where gcd is the greatest common divisor. If $d_i = 1$ then the state i is said to be **aperiodic**.

Irreducible Markov chains have the property that all states have the same period. A Markov chain is called **aperiodic** if some (and hence all) states are aperiodic.

Recurrent/Positive Recurrent

Definition

Let τ_{ii} be the time of the first return to state i

$$\tau_{ii} \quad = \quad \min\{t > 0 : X_t = i \,|\, X_0 = i\}$$

A state i is **recurrent** *if $P(\tau_{ii} < \infty) = 1$ and* **positive recurrent** *if $E(\tau_{ii}) < \infty$.*

Thus, a state i is recurrent if the chain will return to state i with probability 1 and positive recurrent if, with probability 1, it will return in a finite time. An irreducible Markov chain is positive recurrent if some (and hence all) states i are positive recurrent.

Ergodic

Definition

A state is said to be **ergodic** *if it is aperiodic and positive recurrent. A Markov chain is ergodic if all of its states are ergodic.*

Stationary

Definition

A distribution π is said to be a **stationary distribution** *of a Markov chain with transition probabilities $P_{ij} = P(X_t = j \,|\, X_{t-1} = i)$, if*

$$\sum_{i \in S} \pi_i P_{ij} \;=\; \pi_j \;\; \forall j \in S$$

where S denotes the state space.

In matrix notation if P is the matrix of transition probabilities and π the vector with ith entry π_i then the stationary distribution satisfies

$$\pi \;=\; \pi P$$

Existence and uniqueness

Theorem

Each irreducible and aperiodic Markov chain has a unique stationary distribution π*.*

Convergence

Theorem

Let X_t be an irreducible and aperiodic Markov chain with stationary distribution π and arbitrary initial value $X_0 = x_0$. Then

$$P(X_t = x \,|\, X_0 = x_0) \quad \rightarrow \quad \pi(x)$$

as $t \rightarrow \infty$.

Ergodic

Theorem

Let X_t be an ergodic Markov chain with limiting distribution π. If $E\{g(X) \,|\, X \sim \pi(x)\} < \infty$ then the sample mean converges to the expectation of $g(X)$ under π,

$$P\left\{\frac{1}{N}\sum_{i=1}^{N} g(X_i) \rightarrow E\{g(X) \,|\, X \sim \pi(x)\}\right\} \;=\; 1.$$

Consequence of these theorems

If we can construct an ergodic Markov chain θ_t *which has the posterior distribution* $f(\theta \,|\, x)$ *as the stationary distribution* $\pi(\theta)$ *then, starting from an initial point* θ_0*, if we run the Markov chain for long enough, we will sample from the posterior.*

- *for large* t, $\theta_t \sim \pi(\theta) = f(\theta \,|\, x)$
- *for each* $s > t$, $\theta_s \sim \pi(\theta) = f(\theta \,|\, x)$
- *the ergodic averages converge to the desired expectations under the target distribution.*