MA40189 - Question Sheet Four

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Set: Problems class, Thursday 7th March 2019.
Due in: Problems class, Thursday 14th March 2019. If you are unable to make the problems class you should hand it in to me either in lectures or my office, 4W4.10.

Task: Attempt questions 1-2. Questions 3-5 are extra questions which may be discussed in the problems class.

1. Let $X_1, \ldots, X_n$ be exchangeable so that the $X_i$ are conditionally independent given a parameter $\theta$. For each of the distributions
   
   $(a)$ $X_i | \theta \sim Bern(\theta);$  
   $(b)$ $X_i | \theta \sim N(\mu, \theta)$ with $\mu$ known;  
   $(c)$ $X_i | \theta \sim Maxwell(\theta)$, the Maxwell distribution with parameter $\theta$ so that
   
   $$f(x_i | \theta) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \theta^{\frac{3}{2}} x_i^2 \exp\left\{ -\frac{\theta x_i^2}{2} \right\}, \quad x_i > 0$$
   
   and $E(X_i | \theta) = 2\sqrt{\frac{2}{\pi} \theta}$, $Var(X_i | \theta) = \frac{3\pi - 8}{2\pi \theta}$; answer the following questions.
   
   i. Show that $f(x_i | \theta)$ belongs to the 1-parameter exponential family and for $X = (X_1, \ldots, X_n)$ state the sufficient statistic for learning about $\theta$.
   
   ii. By viewing the likelihood as a function of $\theta$, which generic family of distributions (over $\theta$) is the likelihood a kernel of?
   
   iii. By first finding the corresponding posterior distribution for $\theta$ given $x = (x_1, \ldots, x_n)$, show that this family of distributions is conjugate with respect to the likelihood $f(x | \theta)$.

2. Let $X_1, \ldots, X_n$ be exchangeable so that the $X_i$ are conditionally independent given a parameter $\theta$. Suppose that $X_i | \theta$ is geometrically distributed with probability density function
   
   $$f(x_i | \theta) = (1 - \theta)^{x_i-1}\theta, \quad x_i = 1, 2, \ldots$$
   
   (a) Show that $f(x | \theta)$, where $x = (x_1, \ldots, x_n)$, belongs to the 1-parameter exponential family. Hence, or otherwise, find the conjugate prior distribution and corresponding posterior distribution for $\theta$.
   
   (b) Show that the posterior mean for $\theta$ can be written as a weighted average of the prior mean of $\theta$ and the maximum likelihood estimate, $\bar{x}^{-1}$.
(c) Suppose now that the prior for $\theta$ is instead given by the probability density function
\[
f(\theta) = \frac{1}{2B(\alpha + 1, \beta)} \theta^\alpha (1 - \theta)^{\beta - 1} + \frac{1}{2B(\alpha, \beta + 1)} \theta^{\alpha - 1} (1 - \theta)^\beta,
\]
where $B(\alpha, \beta)$ denotes the Beta function evaluated at $\alpha$ and $\beta$. Show that the posterior probability density function can be written as
\[
f(\theta \mid x) = \lambda f_1(\theta) + (1 - \lambda) f_2(\theta)
\]
where
\[
\lambda = \frac{(\alpha + n)\beta}{(\alpha + n)\beta + (\beta - n + \sum_{i=1}^{n} x_i)\alpha}
\]
and $f_1(\theta)$ and $f_2(\theta)$ are probability density functions.

3. Let $X_1, \ldots, X_n$ be exchangeable so that the $X_i$ are conditionally independent given a parameter $\theta$. Suppose that $X_i \mid \theta$ is distributed as a double-exponential distribution with probability density function
\[
f(x_i \mid \theta) = \frac{1}{2\theta} \exp\left\{-\frac{|x_i|}{\theta}\right\}, \quad -\infty < x_i < \infty
\]
for $\theta > 0$.

(a) Find the conjugate prior distribution and corresponding posterior distribution for $\theta$ following observation of $x = (x_1, \ldots, x_n)$.

(b) Consider the transformation $\phi = \theta^{-1}$. Find the posterior distribution of $\phi \mid x$.

4. Let $X_1, \ldots, X_n$ be a finite subset of a sequence of infinitely exchangeable random quantities with joint density function
\[
f(x_1, \ldots, x_n) = n! \left(1 + \sum_{i=1}^{n} x_i\right)^{-(n+1)}
\]
Show that they can be represented as conditionally independent and exponentially distributed.

5. Let $X_1, \ldots, X_n$ be exchangeable so that the $X_i$ are conditionally independent given a parameter $\theta$. Suppose that $X_i \mid \theta$ is distributed as a Poisson distribution with mean $\theta$.

(a) Show that, with respect to this Poisson likelihood, the gamma family of distributions is conjugate.

(b) Interpret the posterior mean of $\theta$ paying particular attention to the cases when we may have weak prior information and strong prior information.

(c) Suppose now that the prior for $\theta$ is given hierarchically. Given $\lambda$, $\theta$ is judged to follow an exponential distribution with mean $\frac{1}{\lambda}$ and $\lambda$ is given the improper distribution $f(\lambda) \propto 1$ for $\lambda > 0$. Show that
\[
f(\lambda \mid x) \propto \frac{\lambda}{(n + \lambda)^{n+x+1}}
\]
where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$. 2