12/2/2021. PROBLEMS CLASS 1

For a random variable X with density $f(x)$ if

$$f(x) = c\,g(x)$$

where c is a constant not depending upon x then $g(x)$ is a KERNEL of the density.

$$\text{e.g. } f(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$\propto \underbrace{\theta^{\alpha-1}(1-\theta)^{\beta-1}}_{\text{A kernel of Beta}(\alpha,\beta)}$$

((Compare Q1 of the homework))

4. $\theta \sim \text{Beta}(4,4)$ $\quad X \mid \theta \sim \text{Bin}(10, x)$. Observe $X < 3$.

$f(\theta \mid x<3)$

$$f(\theta \mid x) \propto \mathbb{P}(X < 3 \mid \theta) f(\theta)$$

$$= \left\{ \sum_{x=0}^{2} \binom{10}{x} \theta^{x} (1-\theta)^{10-x} \right\} \frac{1}{B(4,4)} \underbrace{\theta^{4-1}(1-\theta)^{4-1}}_{\text{keep this in the form of the Beta kernel}}$$

Only interested in functions of θ. Absorb any constant multipliers in $\mathbb{P}(X<3 \mid \theta) f(\theta)$ into the constant of proportionality. Then,

$$f(\theta \mid x) \propto \left\{ \sum_{x=0}^{2} \binom{10}{x} \theta^{x} (1-\theta)^{10-x} \right\} \theta^{4-1}(1-\theta)^{4-1}$$

$$= \sum_{x=0}^{2} \binom{10}{x} \underbrace{\theta^{(4+x)-1}(1-\theta)^{(14-x)-1}}_{\text{Recognise this as a kernel of Beta}(4+x,\,14-x)}$$

[Compare this with question 2 of the homework]

[Remember: if $\theta \sim \text{Beta}(\alpha,\beta)$, $X \mid \theta \sim \text{Bin}(n,\theta)$ and observe $X = x$ then $\theta \mid x \sim \text{Beta}(\alpha + x, \beta + n - x)$]

successes   # FAILURES

Hence,
$$f(\theta|x) = k\left[\sum_{x=0}^{2}\binom{10}{x}\theta^{(4+x)-1}(1-\theta)^{(14-x)-1}\right]$$

As $\int_0^1 f(\theta|x)\,d\theta = 1$ then:
$$k = \left[\int_0^1 \sum_{x=0}^{2}\binom{10}{x}\theta^{(4+x)-1}(1-\theta)^{(14-x)-1}\,d\theta\right]^{-1}$$

$$= \left[\sum_{x=0}^{2}\binom{10}{x}\int_0^1 \theta^{(4+x)-1}(1-\theta)^{(14-x)-1}\,d\theta\right]^{-1}$$

as the integrand is a kernel of Beta$(4+x, 14-x)$ then I know the value of this integral

[Compare with Q3 of the homework].

$$= \left[\sum_{x=0}^{2}\binom{10}{x}B(4+x, 14-x)\right]^{-1}$$

Thus, $$k^{-1} = \sum_{x=0}^{2}\binom{10}{x}B(4+x, 14-x)$$

$$= B(4,14) + 10\,B(5,13) + 45\,B(6,12)$$

$$= \frac{\Gamma(4)\Gamma(14)}{\Gamma(18)} + 10\,\frac{\Gamma(5)\Gamma(13)}{\Gamma(18)} + 45\,\frac{\Gamma(6)\Gamma(12)}{\Gamma(18)}$$

$[\,\Gamma(z+1) = z\,\Gamma(z)\,]$

$$= \frac{\Gamma(4)\Gamma(12)}{\Gamma(18)}\left[\overbrace{(13\times 12)}^{156} + \overbrace{10\,(4\times 12)}^{480} + \overbrace{45\,(5\times 4)}^{900}\right]$$

$$= 1536\,\frac{\Gamma(4)\Gamma(12)}{\Gamma(18)}$$

i.e. $$f(\theta|x) = \frac{\Gamma(18)}{1536\,\Gamma(4)\Gamma(12)} \left[\sum_{x=0}^{2} \binom{10}{x} \theta^{(4+x)-1} (1-\theta)^{(14-x)-1} \right]$$

[Note: $$f(\theta|x) = \frac{\Gamma(18)}{1536\,\Gamma(4)\Gamma(12)} \left[\sum_{x=0}^{2} \binom{10}{x} \beta(4+x, 14-x) \underbrace{\frac{1}{\beta(4+x, 14-x)} \theta^{(4+x)-1} (1-\theta)^{(14-x)-1}}_{\text{pdf of Beta }(4+x,\, 14-x)} \right]$$

Let $f_x(\theta)$ be the pdf of Beta $(4+x, 14-x)$ then:

$$f(\theta|x) = \frac{\Gamma(18)}{1536\,\Gamma(4)\Gamma(12)} \left[\sum_{x=0}^{2} \binom{10}{x} \beta(4+x, 14-x) f_x(\theta) \right]$$

i.e. $$f(\theta|x) = \alpha_0 f_0(\theta) + \alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$$

where: $f_x(\theta)$ is the pdf of Beta $(4+x, 14-x)$ and $\alpha_x \geq 0$ $(x = 0, 1, 2)$ and $\alpha_0 + \alpha_1 + \alpha_2 = 1$. This is a MIXTURE DISTRIBUTION.]

b). $$E(\theta|X) = \int_0^1 \theta \frac{\Gamma(18)}{1536\,\Gamma(4)\Gamma(12)} \left[\sum_{x=0}^{2} \binom{10}{x} \theta^{(4+x)-1} (1-\theta)^{(14-x)-1} \right] d\theta$$

$$= k \sum_{x=0}^{2} \binom{10}{x} \int_0^1 \underbrace{\theta^{(5+x)-1} (1-\theta)^{(14-x)-1}}_{\text{kernel of Beta }(5+x,\, 14-x)} d\theta$$

$$= k \sum_{x=0}^{2} \binom{10}{x} \beta(5+x, 14-x)$$

$$= k \left[\frac{\Gamma(15)\Gamma(14)}{\Gamma(19)} + 10\,\frac{\Gamma(6)\Gamma(13)}{\Gamma(19)} + 45\,\frac{\Gamma(7)\Gamma(12)}{\Gamma(19)} \right]$$

$$= \frac{2106}{1536} \frac{\cancel{\Gamma(18)}}{\cancel{\Gamma(4)}\cancel{\Gamma(12)}} \times \frac{\overset{4}{\cancel{\Gamma(5)}}\cancel{\Gamma(12)}}{\cancel{\Gamma(19)}\,18} = \frac{39}{128}.$$