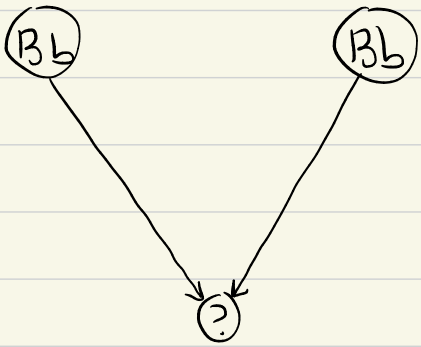
5/2/2021. PROBLEMS CLASS 0.

Heterozygous (Bb) Homozygous (BB or Bb)
brown (bb) black (BB or Bb).

Generation 0: Parents.



G0: Both parents are Bb.

Inherited gene is equally likely to be B or b.

a). Generation 1: offspring.

Possibilities: G1 - BB offspring is BB $\}$ Black
G1 - Bb offspring is Bb
G1 - bb offspring is bb $\}$ Brown

Interested in how these probabilities change with data.

$$\mathbb{P}(G1-BB \mid G0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$\mathbb{P}(G1-Bb \mid G0) = 2\left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{2}$$

$$\mathbb{P}(G1-bb \mid G0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Hence, $\mathbb{P}(G1-\text{black} \mid G0) = \mathbb{P}(G1-BB \mid G0) + \mathbb{P}(G1-Bb \mid G0)$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Similarly, $\mathbb{P}(\text{G1-brown} \mid \text{G0}) = \mathbb{P}(\text{G1-bb} \mid \text{G0}) = \frac{1}{4}$.

Learn that "G1-black" occurs.

$$\text{Current knowledge: } \overbrace{\text{G1-black}}^{\text{New data}}, \text{G0}$$

Update probabilities for G1-BB, G1-Bb, G1-bb given this new information.

$$\overbrace{\mathbb{P}(\text{G1-BB} \mid \text{G1-black}, \text{G0})}^{\text{"Posterior" for G1-BB}} = \frac{\mathbb{P}(\text{G1-BB}, \text{G1-black} \mid \text{G0})}{\mathbb{P}(\text{G1-black} \mid \text{G0})}$$

$$= \frac{\overbrace{\mathbb{P}(\text{G1-black} \mid \text{G1-BB}, \text{G0})}^{\text{"Likelihood" probability of data/model}} \overbrace{\mathbb{P}(\text{G1-BB} \mid \text{G0})}^{\text{"Prior" for G1-BB}}}{\mathbb{P}(\text{G1-black} \mid \text{G0})}$$

$$= \frac{1 \times 1/4}{3/4} = \frac{1}{3}.$$

$$\overbrace{\mathbb{P}(\text{G1-Bb} \mid \text{G1-black}, \text{G0})}^{\text{Posterior for G1-Bb}} = \frac{\overbrace{\mathbb{P}(\text{G1-black} \mid \text{G1-Bb}, \text{G0})}^{\text{Likelihood}} \overbrace{\mathbb{P}(\text{G1-Bb} \mid \text{G0})}^{\text{Prior for G1-Bb}}}{\mathbb{P}(\text{G1-black} \mid \text{G0})}$$

$$= \frac{1 \times 1/2}{3/4} = \frac{2}{3}$$

[$\mathbb{P}(\text{G1-bb} \mid \text{G1-black}, \text{G0}) = 0$ as $\mathbb{P}(\text{G1-black} \mid \text{G1-bb}, \text{G0}) = 0$]

Hence,

$$\mathbb{P}(\text{G1-homozygous} \mid \text{G1-black}, \text{G0}) = \mathbb{P}(\text{G1-BB} \mid \text{G1-black}, \text{G0}) = 1/3$$

$$\mathbb{P}(\text{G1-heterozygous} \mid \text{G1-black}, \text{G0}) = \mathbb{P}(\text{G1-Bb} \mid \text{G1-black}, \text{G0}) = 2/3$$

"Parameter"	G1-BB	G1-Bb	G1-bb
"Prior"	$1/4$	$1/2$	$1/4$
"Likelihood"	1	1	0
Prior x Likelihood	$1/4$	$1/2$	0
Posterior	$\frac{1/4}{1/4+1/2+0} = \frac{1}{3}$	$\frac{1/2}{1/4+1/2+0} = \frac{2}{3}$	$\frac{0}{1/4+1/2+0} = 0$

[NB. G1-black rules out G1-bb].

b). Generation 2: 7BO = "7 black offspring"

"G1-black" mated with bb = "G1"

$$\mathbb{P}(G1\text{-}BB \mid G1, G0) = \mathbb{P}(G1\text{-}BB \mid G1\text{-black}, G0)$$
$$\mathbb{P}(G1\text{-}Bb \mid G1, G0) = \mathbb{P}(G1\text{-}Bb \mid G1\text{-black}, G0)$$

These are my new prior probabilities.

New data is 7BO in G2. Compute the likelihood for these.

new data parameter old data

$$\mathbb{P}(7BO \mid G1\text{-}BB, G1, G0) = 1$$
$$\mathbb{P}(7BO \mid G1\text{-}Bb, G1, G0) = \left(\frac{1}{2}\right)^7$$

New posteriors.

Posterior $\propto$ Likelihood x Prior

$$\mathbb{P}(G1\text{-}BB \mid 7BO, G1, G0) \propto \mathbb{P}(7BO \mid G1\text{-}BB, G1, G0)\,\mathbb{P}(G1\text{-}BB \mid G1, G0)$$
$$= 1 \times 1/3 = 1/3.$$

$$\mathbb{P}(G1\text{-}Bb \mid 7BO, G1, G0) \propto \mathbb{P}(7BO \mid G1\text{-}Bb, G1, G0)\,\mathbb{P}(G1\text{-}Bb \mid G1, G0)$$
$$= \left(\frac{1}{2}\right)^7 \times \frac{2}{3} = \left(\frac{1}{2}\right)^6 \times \frac{1}{3}$$

Thus, normalising,

$$P(G1\text{-}BB \mid 7BO, G1, G0) = \frac{1/3}{1/3 + (1/2)^6\, 1/3} = \frac{2^6}{2^6+1} = \frac{64}{65}$$

$$P(G1\text{-}Bb \mid 7BO, G1, G0) = \frac{(1/2)^6\, 1/3}{1/3 + (1/2)^6\, 1/3} = \frac{1}{2^6+1} = \frac{1}{65}.$$

c). Sequential updates.

After n BO in G2, given G1, G0	G1-BB	G1-Bb
"Prior"	$\dfrac{2^{n-1}}{2^{n-1}+1}$	$\dfrac{1}{2^{n-1}+1}$
Likelihood (for a single offspring)	1	$\dfrac{1}{2}$
Prior $\times$ Likelihood	$\dfrac{2^{n-1}}{2^{n-1}+1}$	$\dfrac{1}{2(2^{n-1}+1)}$
Posterior (normalising)	$\dfrac{2^{n-1}}{2^{n-1}+1/2} = \dfrac{2^n}{2^n+1}$	$\dfrac{1/2}{2^{n-1}+1/2} = \dfrac{1}{2^n+1}$

The outcome is the same if the data is added sequentially to when it is added on a whole.

d) Predictive distribution.

$$P(G2\text{-}8BO \mid G2\text{-}7BO, G1, G0)$$

$$\begin{aligned}
&= P(G2\text{-}8BO \mid G1\text{-}BB, G2\text{-}7BO, G1, G0)\,P(G1\text{-}BB \mid G2\text{-}7BO, G1, G0) \\
&+ P(G2\text{-}8BO \mid G1\text{-}Bb, G2\text{-}7BO, G1, G0)\,P(G1\text{-}Bb \mid G2\text{-}7BO, G1, G0) \\
&= \left(1 \times \frac{64}{65}\right) + \left(\frac{1}{2} \times \frac{1}{65}\right) = \frac{129}{130}.
\end{aligned}$$