IN ESSENCE, EXCHANGEABILITY CAPTURES THE NOTION THAT ONLY THE VALUES MATTER AND NOT THE ORDER IN WHICH THE OBSERVATIONS OCCUR. THE LABELS ARE UNINFORMATIVE.

NB. FINITELY EXCHANGEABLE IMPLIES THAT

\[ f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \]

FOR ALL \( m \leq n \).

E.g., \( f_x(x) = \int f_{x,y}(x,y) \, dy = \int f_{x,y}(y,x) \, dy = f_y(x) \]

EXCHANGEABILITY IS A STRONGER STATEMENT THAN IDENTICALLY DISTRIBUTED (JOINTS INCLUDED) BUT WEAKER THAN INDEPENDENCE.

IID OBSERVATIONS ARE EXCHANGEABLE:

\[ f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) \]

BUT EXCHANGEABLE OBSERVATIONS NEED NOT BE INDEPENDENT.

EXAMPLE.

\( X_1 \) AND \( X_2 \) ARE EXCHANGEABLE RVs WITH JOINT DENSITY

\[ f(x_1, x_2) = 3 \left( x_1^2 + x_2^2 \right) \quad 0 < x_1 < 1 \]

\[ 0 < x_2 < 1 \]

HOWEVER, THEY ARE NOT INDEPENDENT.

DEFINITION (INFINITE EXCHANGEABILITY).

THE INFINITE SEQUENCE OF RVs \( X_1, X_2, \ldots \) ARE JUDGED TO BE INFINITELY EXCHANGEABLE IF EVERY FINITE SUBSEQUENCE IS JUDGED (FINITELY) EXCHANGEABLE.
A natural question of interest is whether every finitely exchangeable sequence can be embedded into an infinitely exchangeable sequence?

Example

Suppose that $X_1, X_2, X_3$ are three exchangeable events (i.e., 0-1 RVs) with:

$$P(X_1 = 0, X_2 = 1, X_3 = 1) = P(X_1 = 1, X_2 = 0, X_3 = 1) = \frac{1}{3},$$

with all other combinations having probability 0.

Is there an $X_4$ such that $X_1, X_2, X_3$ and $X_4$ are exchangeable?

If so, then, for example,

$$P(X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0) = P(X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1).$$

Now,

$$P(X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0) = P(X_1 = 0, X_2 = 1, X_3 = 1) - P(X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 1) = \frac{1}{3} - P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 0)$$

However, $P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 0) \leq P(X_1 = 1, X_2 = 1, X_3 = 1) = 0$

$$\Rightarrow P(X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0) = \frac{1}{3}.$$
However,
\[ P(X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1) \neq P(X_1 = 0, X_2 = 0, X_3 = 1) = 0. \]

This contradiction shows that \( X_1, X_2, X_3, X_4 \) are not exchangeable.

A finitely exchangeable sequence cannot necessarily be embedded into a larger finitely exchangeable sequence let alone an infinitely exchangeable one.

**Theorem (De Finetti's Representation Theorem for 0-1 RVs)**

Let \( X_1, X_2, \ldots \) be a sequence of infinitely exchangeable 0-1 RVs (i.e. events) then the joint distribution of \( X_1, \ldots, X_n \) has an integral representation of the form

\[
f(x_1, \ldots, x_n) = \int_0^1 \left\{ \prod_{i=1}^n \theta^x_i (1-\theta)^{1-x_i} \right\} f(\theta) \, d\theta
\]

where \( y_n = \sum_{i=1}^n x_i \) and \( \theta = \lim_{n \to \infty} \frac{y_n}{n} \).

The interpretation of this theorem is of profound significance:

1. Conditional on a RV \( \theta \), the \( X_i \) are judged to be independent Bernoulli trials

\[
f(x_i | \theta) = \theta^x_i (1-\theta)^{1-x_i}
\]

\[
f(x_1, \ldots, x_n | \theta) = \prod_{i=1}^n \theta^x_i (1-\theta)^{1-x_i} = \theta^{y_n} (1-\theta)^{n-y_n}
\]

2. \( \theta \) is itself assigned a probability distribution \( f(\theta) \).
(3) \( \theta = \lim_{n \to \infty} \frac{y_n}{n} \) so that \( f(\theta) \) represents beliefs about the limiting value of the mean of the \( x_\cdot \). Conditional upon \( \theta \), \( x_1, \ldots, x_n \) are a random sample from a Bernoulli with parameter \( \theta \) generating a parameterised joint sampling distribution

\[
f(x_1, \ldots, x_n | \theta) = \prod_{i=1}^{n} \theta^{x_i} (1 - \theta)^{1-x_i}
\]

where \( \theta \) is assigned a prior distribution \( f(\theta) \).

This provides a justification for the Bayesian approach (combining a prior and a likelihood) [for events: generalisation follows].

\[ \text{Heze} \]

NB. \( \theta \) is provided with a formal definition and the sentence "the true value of \( \theta \)" has a well-defined meaning.