LAST TIME: \( \Theta \sim N(\mu_0, \sigma_0^2) \), \( X \mid \Theta \sim N(\Theta, \sigma^2) \).

\[
p(\Theta \mid X) \propto \exp \left\{ -\frac{1}{2} \left( \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2} \right) \left[ \Theta - 2 \left( \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2} \right)^{-1} \left( \frac{\mu_0}{\sigma_0^2} + \frac{\mu}{\sigma^2} \right) \right] \right\}
\]

*This is a kernel of a normal distribution so that:*

\[\Theta \mid X \sim N(\mu, \sigma_1^2)\]

*WHERE \( \sigma_1^2 = \left( \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2} \right)^{-1}\) i.e. \( \frac{1}{\sigma^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2} \)

*Posterior = Prior + Data

*Precision = Precision + Precision

*\[\mu = \left( \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2} \right)^{-1} \left( \frac{\mu_0}{\sigma_0^2} + \frac{\mu}{\sigma^2} \right) = \alpha \mu_0 + (1 - \alpha) \mu\]*

*\[\alpha = \frac{1/\sigma_0^2}{1/\sigma_0^2 + 1/\sigma^2}\]*

*Weighted average of prior mean \( \mu_0 \) and data \( \mu \), weighted according to their corresponding precisions.

*(HIGH CONFIDENCE = LOW VARIANCE = HIGH PRECISION)*

i.e. STRONG PRIOR \(\Rightarrow\) LARGE \(1/\sigma_0^2\)

**INCREASE WEIGHT ON \( \mu_0 \)**

WEAK PRIOR \(\Rightarrow\) SMALL \(1/\sigma_0^2\)

**INCREASE WEIGHT ON \( \mu \)**

*GIVEN A NORMAL LIKELIHOOD (WITH KNOWN VARIANCE), THE NORMAL DISTRIBUTION IS A CONJUGATE FAMILY.*
1.5 USING THE POSTERIOR FOR INFERENCE.

POSTERIOR BELIEFS (CAPTURED BY \( f(\theta | x) \)), A HOLE DISTRIBUTION.
TYPICALLY WE WANT TO SUMMARISE THIS DISTRIBUTION.

E.G. GRAPHS AND PLOTS OF SHAPE, MEASURES OF LOCATION
(MEAN, MEDIAN, MODE), MEASURES OF DISPERSION (VARIANCE,
QUARTILES). REGION THAT CAPTURES MOST OF THE VALUES OF \( \theta \).

WE FOCUS ON REGIONS. IN THE FOLLOWING ASSUME \( \theta \) IS UNIVARIATE
(RESULTS AND DEFINITIONS SIMPLY GENERALISE FOR A MULTIVARIATE
WHERE APPROPRIATE).

1.5.1 CREDIBLE INTERVALS AND HIGHEST DENSITY REGIONS.

(CREDIBLE INTERVALS (OR POSTERIOR INTERVALS) ARE THE BAYESIAN
ANALOGUE OF CONFIDENCE INTERVALS.

A 100(1-\( \alpha \))% CI IS INTERPRETED AS MEANING THAT WITH
A LARGE NUMBER OF REPEATED SAMPLES, 100(1-\( \alpha \))% OF THE
INTERVALS WOULD CONTAIN THE TRUE VALUE OF \( \theta \).

E.G. CONFIDENCE INTERVAL \((\theta^*_L(x), \theta^*_U(x))\) WHERE:

\[
P(\theta^*_L(x) < \theta < \theta^*_U(x) | x) = 1 - \alpha
\]

RANDOM \( \Rightarrow \) FIXED \( \Rightarrow \) RANDOM

I.E. REALISATION OF A RANDOM INTERVAL, CONSTRUCTED
USING \( f(x | \theta) \).

A 100(1-\( \alpha \))% CREDIBLE INTERVAL IS AN ACTUAL INTERVAL
THAT CONTAINS \( \theta \) WITH PROBABILITY 1-\( \alpha \).
**Definition (credible interval).**

A 100(1-\(\alpha\))% **credible interval** \((\theta_L, \theta_U)\) is an interval within which 100(1-\(\alpha\))% of the posterior distribution lies.

\[
P(\theta_L < \theta < \theta_U | X) = \int_{\theta_L}^{\theta_U} f(\theta | X) \, d\theta = 1 - \alpha.
\]

Fixed  Random  Fixed

Note that there are infinitely many such intervals, so we typically fix the interval by specifying the tail probability:

\[
e.g., \quad P(\theta < \theta_L | X) = \frac{\alpha}{2} = P(\theta > \theta_U | X).
\]

We can construct similar intervals for any distribution over a univariate RV in particular for the prior when the intervals are often termed **prior credible intervals**.

**Example:** Tossing coins and drawing pins; 95% intervals.

<table>
<thead>
<tr>
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<th>Prior</th>
<th>Prior Cred</th>
<th>Post</th>
<th>Post Cred</th>
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<tr>
<td><strong>Coins</strong></td>
<td>Beta (20, 20)</td>
<td>(0.3478, 0.6522)</td>
<td>Beta (24,26)</td>
<td>(0.3442, 0.6173)</td>
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<tr>
<td>(Strong)</td>
<td></td>
<td><strong>Middle 0.5</strong></td>
<td></td>
<td><strong>Middle 0.48075</strong></td>
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<td></td>
<td>←0.3044→</td>
<td></td>
<td>←0.2931→</td>
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<tr>
<td><strong>Pins</strong></td>
<td>Beta (2, 2)</td>
<td>(0.0950, 0.9057)</td>
<td>Beta (6, 8)</td>
<td>(0.1922, 0.6842)</td>
</tr>
<tr>
<td></td>
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<td><strong>Middle 0.5</strong></td>
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<td><strong>Middle 0.4383</strong></td>
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<tr>
<td></td>
<td></td>
<td>←0.8107→</td>
<td></td>
<td>←0.491→</td>
</tr>
</tbody>
</table>

\[\theta_L = \text{quantile}(0.025, \alpha, \beta), \quad \theta_U = \text{quantile}(0.975, \alpha, \beta).\]