

9/2/2021. LECTURE 3.

1.3 SEQUENTIAL DATA ANALYSIS.

Suppose we have two sources of data x and y . We can add the data sequentially.

$$f(\theta | x, y) \propto f(x, y | \theta) f(\theta)$$

\swarrow likelihood is of the form data | model

Now,

$$f(x, y | \theta) = f(y | x, \theta) f(x | \theta)$$

$\underbrace{\hspace{10em}}$ As a function of θ , this is proportional to $f(\theta | x)$

so that

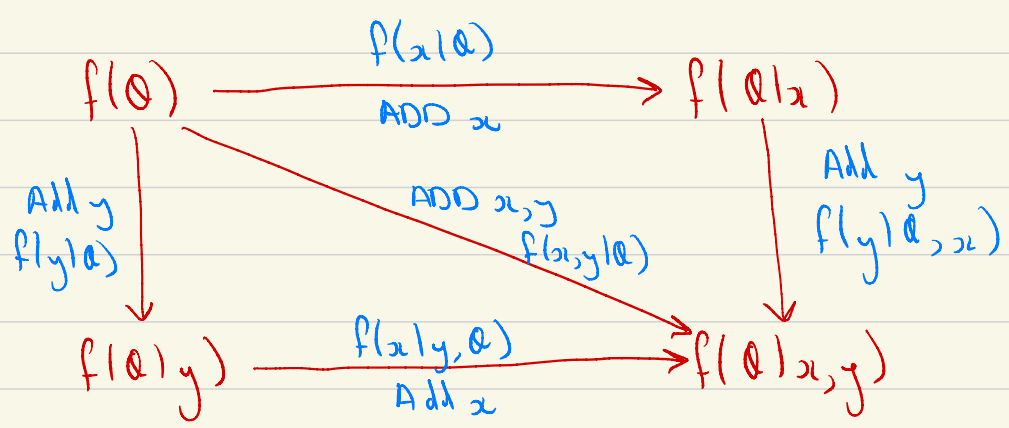
$$f(\theta | x, y) \propto f(y | x, \theta) f(x | \theta) f(\theta)$$
$$\propto f(y | x, \theta) f(\theta | x)$$

\swarrow likelihood for the data y given observed x (and the model) \swarrow update of θ given x

[This is thus in the form "Posterior" \propto "Likelihood" \times "Prior"]

We can first update by x and then by y :

Prior	likelihood	Posterior	
$f(\theta)$	$f(x \theta)$	$f(\theta x)$	Added the data x
$f(\theta x)$	$f(y x, \theta)$	$f(\theta x, y)$	Added the data x, y .



Note that if X and Y are conditionally independent given θ [$(X \perp\!\!\!\perp Y) | \theta$] then:

$$f(x, y | \theta) = f(x | \theta) f(y | \theta)$$

$$[\text{or } f(y | x, \theta) = f(y | \theta)]$$

Thus,

$$f(\theta | x, y) \propto f(y | \theta) f(x | \theta) f(\theta)$$
$$\propto \underbrace{f(y | \theta) f(x | \theta)}_{\text{Multiply likelihoods}} f(\theta)$$

[analogous to independent observations in the classical model].

1.4 CONJUGATE BAYESIAN UPDATES.

EXAMPLE: BETA-BINOMIAL.

Suppose $X | \theta \sim \text{Bin}(n, \theta)$ and we wish to specify a prior distribution for θ . Consider $\theta \sim \text{Beta}(\alpha, \beta)$ for $\alpha, \beta > 0$ known

[i.e. we specify some numerical values for these. An extension would be to treat α, β as random variables and specify prior distributions for those: idea of hierarchical model].

Thus, for $0 \leq \theta \leq 1$

$$f(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

when the **BETA FUNCTION** $B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}$

and $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ is the **GAMMA FUNCTION**.

Note that $\Gamma(z+1) = z \Gamma(z)$ and $\Gamma(1) = 1$

[If $z \in \mathbb{Z}^+$ then $\Gamma(z+1) = z!$]

[Note: we will typically utilise the Gamma function for positive reals rather than integers so you should use $\Gamma(z+1) = z \Gamma(z)$ rather than $\Gamma(z+1) = z!$. The former is the general case]

Similarly,

$$B(\alpha, \beta) = \int_0^1 a^{\alpha-1} (1-a)^{\beta-1} da.$$

$$\text{Note: } E(a) = \int_0^1 a \frac{1}{B(\alpha, \beta)} a^{\alpha-1} (1-a)^{\beta-1} da$$

$$= \int_0^1 \frac{1}{B(\alpha, \beta)} a^{(\alpha+1)-1} (1-a)^{\beta-1} da$$

$$= \frac{1}{B(\alpha, \beta)} \int_0^1 a^{(\alpha+1)-1} (1-a)^{\beta-1} da$$

$$= \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+1) \Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)}$$

$$= \frac{\alpha \Gamma(\alpha)}{(\alpha+\beta) \Gamma(\alpha+\beta)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} = \frac{\alpha}{\alpha+\beta}$$

$$[E[1-a] = 1 - E(a) = \frac{\beta}{\alpha+\beta}]$$

Similarly,

$$E(a^2) = \int_0^1 a^2 \frac{1}{B(\alpha, \beta)} a^{\alpha-1} (1-a)^{\beta-1} da = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$