We now generate a Gibbs sequence of RVs.

1. Initial value $\theta^{(0)} = 0, 1, \ldots, 3$, $0 < \theta^{(0)}_2 < 1$ (or sample from $\pi(\theta^0)$).

2. At time $t$,

   - Obtain $\theta^{(t)}_1$ by sampling from $\text{Bin}(n, \theta^{(t-1)}_2)$, the distribution of $\theta^{(t-1)}_1 | \theta^{(t-1)}_2$.
     - $\text{Sample could be direct or by M-H.}$
   - Obtain $\theta^{(t)}_2$ by sampling from $\text{Beta}(\alpha^{(t)}_1 + n - \theta^{(t)}_1 + \beta, \alpha^{(t)}_2)$, the distribution of $\theta^{(t)}_2 | \theta^{(t)}_1$.

3. Repeat Step 2.

The sequence $\theta^{(0)}_1, \theta^{(0)}_2, \theta^{(1)}_1, \theta^{(1)}_2, \ldots$ is such that the distribution of $(\theta^{(k)}_1, \theta^{(k)}_2)$ tends to $\pi(\theta_1, \theta_2) = f(\theta_1, \theta_2)$ as $k \to \infty$.

The distribution of $\theta^{(k)}_1$ tends to $\pi(\theta_1) = f(\theta_1)$ and that of $\theta^{(k)}_2$ tends to $\pi(\theta_2) = f(\theta_2)$ as $k \to \infty$.

We illustrate this in R. To do so, we find the two marginal distributions.

\[
f(\theta_1) \propto \int_0^1 \binom{n}{\theta_1} \theta_2^{\alpha_1+n-1} (1-\theta_2)^{\alpha_2-\beta-1} d\theta_2
\]

\[
= \binom{n}{\theta_1} \int_0^1 \theta_2^{\alpha_1+n-1} (1-\theta_2)^{\alpha_2-\beta-1} d\theta_2
\]

Kernel of $\text{Beta}(\alpha_1 + n - \theta_1 + \beta, \alpha_2)$

\[i.e., f(\theta_2 | \theta_1).\]
\[ f(\theta_2) = \sum_{\theta_1=0}^{\frac{n}{\alpha_1}} \binom{n}{\alpha_1} \bigg( \frac{\alpha_2}{\alpha_2+\theta_2-1} \bigg)^{\alpha_2+\theta_2-1} \bigg( \frac{1-\theta_2}{\alpha_2+\theta_2-1} \bigg)^{n-\alpha_2-\theta_2} \]

\[ = \frac{n!}{\alpha! (n-\alpha)!} \frac{\Gamma(\alpha_1+1) \Gamma(\alpha_2)}{\Gamma(n+2)} = \frac{1}{n+1} \]

\[ \text{or} \quad \theta_2 \sim \text{Beta}(\alpha_2, \beta) \]

\[ \text{Example.} \]

\[ n = 10, \alpha = 1, \beta = 1, \theta_1^{(0)} = 5, \theta_2^{(0)} = 0.5 \]

\[ \text{Figure 1: gibbs 2 (n=50, m=10, a=1, b=1, start=5, step=0.5)} \]

\[ \text{Figure 2: gibbs 2 (n=1000, m=10, a=1, b=1, start=5, step=0.5)} \]

\[ n=10, \alpha = 2, \beta = 3, \theta_1^{(0)} = 5, \theta_2^{(0)} = 0.5 \]

\[ \text{Figure 3: gibbs 2 (n=50, m=10, a=2, b=3, start=5, step=2=0.5)} \]

\[ \text{Figure 4: gibbs 2 (n=1000, m=10, a=2, b=3, start=5, step=2=0.5)} \]
Figure 1: Fifty iterations from a Gibbs sampler for $x_1 | x_2 \sim Bin(n, x_2)$, $x_2 | x_1 \sim Beta(x_1 + \alpha, n - x_1 + \beta)$ where $n = 10$ and $\alpha = \beta = 1$. The marginal distributions are $x_1 \sim Beta - binomial(n, \alpha, \beta)$ and $x_2 \sim Beta(\alpha, \beta)$. For $\alpha = \beta = 1$, $x_1$ is the discrete uniform on $\{0, 1, \ldots, n\}$ and $x_2 \sim U(0, 1)$. 
Figure 2: 1500 iterations from a Gibbs sampler for $x_1 \mid x_2 \sim \text{Bin}(n, x_2)$, $x_2 \mid x_1 \sim \text{Beta}(x_1 + \alpha, n - x_1 + \beta)$ where $n = 10$ and $\alpha = \beta = 1$. The marginal distributions are $x_1 \sim \text{Beta - binomial}(n, \alpha, \beta)$ and $x_2 \sim \text{Beta}(\alpha, \beta)$. For $\alpha = \beta = 1$, $x_1$ is the discrete uniform on $\{0, 1, \ldots, n\}$ and $x_2 \sim U(0, 1)$. 
Figure 3: Fifty iterations from a Gibbs sampler for $x_1 | x_2 \sim Bin(n, x_2)$, $x_2 | x_1 \sim Beta(x_1 + \alpha, n - x_1 + \beta)$ where $n = 10, \alpha = 2$ and $\beta = 3$. The marginal distributions are $x_2 \sim Binomial(n, \alpha, \beta)$ and $x_2 \sim Beta(\alpha, \beta)$. 
Figure 4: 1500 iterations from a Gibbs sampler for $x_1 \mid x_2 \sim Bin(n, x_2)$, $x_2 \mid x_1 \sim Beta(x_1 + \alpha, n - x_1 + \beta)$ where $n = 10$, $\alpha = 2$ and $\beta = 3$. The marginal distributions are $x_1 \sim Beta - binomial(n, \alpha, \beta)$ and $x_2 \sim Beta(\alpha, \beta)$. 


PROBLEMS CLASS MATERIAL: ADOPTION OF Q5 OF QUESTION SHEET EIGHT. VIEWING EACH STEP OF THE GIBBS Sampler AS A M-H MOVE.

\[ q(\ldots, \theta^{(t-1)} \mid \theta^{(t-1)}) = \kappa(\ldots, \theta^{(t-1)} \mid \theta^{(t-1)}) \]

a). MOVE TO \( \theta^{*}_{t+1} = (\theta^{(t)}, \theta^{(t-1)}_{2}) \) WITH PROPOSAL:

\[ q(\theta^{*}_{t+1}, \theta^{(t-1)}) = \kappa(\theta^{(t)}, \theta^{(t-1)}_{2}) \]

\[ q(\theta^{(t-1)} \mid \theta^{*}_{t+1}) = \kappa(\theta^{(t-1)} \mid \theta^{(t-1)}) \]

Now, \( \kappa(\theta^{*}_{t+1}) = \kappa(\theta^{(t)}, \theta^{(t-1)}_{2}) \)

\[ = \kappa(\theta^{(t)} \mid \theta^{(t-1)}_{2}) \kappa(\theta^{(t-1)}) \]

\[ = q(\theta^{*}_{t+1} \mid \theta^{(t-1)}_{2}) \kappa(\theta^{(t-1)}) \]

\[ \kappa(\theta^{(t-1)}) = \kappa(\theta^{(t-1)} \mid \theta^{(t-1)}_{2}) \kappa(\theta^{(t-1)}_{2}) \]

\[ = q(\theta^{*}_{t+1} \mid \theta^{(t-1)}_{2}) \kappa(\theta^{(t-1)}_{2}) \]

Thus, \( \frac{\kappa(\theta^{*}_{t+1})}{\kappa(\theta^{(t-1)})} = \frac{q(\theta^{*}_{t+1} \mid \theta^{(t-1)}_{2})}{\kappa(\theta^{(t-1)}_{2})} \)

\[ \kappa(\theta^{(t-1)}, \theta^{*}_{t+1}) = \min(1, \frac{\kappa(\theta^{*}_{t+1}) q(\theta^{(t-1)} \mid \theta^{*}_{t+1})}{\kappa(\theta^{(t-1)}_{2}) q(\theta^{(t-1)}_{2} \mid \theta^{(t-1)})}) \]

\[ = 1. \]

b). MOVE TO \( \theta^{(t)} = (\theta^{(t)}_{1}, \theta^{(t)}_{2}) \) FROM \( \theta^{*}_{t+1} \) WITH PROPOSAL

\[ q(\theta^{(t)} \mid \theta^{*}_{t+1}) = \kappa(\theta^{(t)}_{2} \mid \theta^{(t)}_{1}) \]


\[ q \left( \hat{\Theta}^{+}_1 \mid \Theta^{(t)} \right) = \pi \left( \Theta^{(t-1)} \mid \Theta^{(t)} \right). \]

Now,
\[ \pi \left( \Theta^{(t)} \right) = \pi \left( \Theta^{(t)} \mid \Theta^{(t-1)} \right) \pi \left( \Theta^{(t-1)} \right) = \pi \left( \Theta^{(t-1)} \mid \Theta^{(t)} \right) \pi \left( \Theta^{(t)} \right) = q \left( \Theta^{(t)} \mid \Theta^{+}_1 \right) \pi \left( \Theta^{(t)} \right) \]
\[ \pi \left( \Theta^{+}_1 \right) = \pi \left( \Theta^{(t)} \mid \Theta^{(t-1)} \right) \pi \left( \Theta^{(t-1)} \right) = \pi \left( \Theta^{(t-1)} \mid \Theta^{(t)} \right) \pi \left( \Theta^{(t)} \right) = q \left( \Theta^{+}_1 \mid \Theta^{(t)} \right) \pi \left( \Theta^{(t)} \right) \]

So that
\[ \frac{\pi \left( \Theta^{(t)} \right)}{\pi \left( \Theta^{+}_1 \right)} = \frac{q \left( \Theta^{(t)} \mid \Theta^{+}_1 \right)}{q \left( \Theta^{+}_1 \mid \Theta^{(t)} \right)} \]

AND \[ \pi \left( \Theta^{+}_1, \Theta^{(t)} \right) = \min \left( 1, \frac{\pi \left( \Theta^{(t)} \right) q \left( \Theta^{+}_1 \mid \Theta^{(t)} \right)}{\pi \left( \Theta^{+}_1 \right) q \left( \Theta^{(t)} \mid \Theta^{+}_1 \right)} \right) = 1. \]

**Sampling Idea.**

\[ \frac{1}{n} \sum_{i=1}^{n} g(\Theta^{(i)}) \Rightarrow E \left( g(\Theta) \mid \Theta^{(t)} \right) \pi(\Theta) \text{ as } n \to \infty \]

\[ \frac{1}{n} \left( \sum_{i=1}^{b} g(\Theta^{(i)}) + \sum_{i=b+1}^{n} g(\Theta^{(i)}) \right) \Rightarrow E \left( g(\Theta) \mid \Theta^{(t)} \right) \pi(\Theta) \]

*Prior to convergence*

*After convergence*

\[ \Rightarrow \frac{1}{N-b} \sum_{c=b+1}^{N} g(\Theta^{(c)}) \Rightarrow E \left( g(\Theta) \mid \Theta^{(t)} \right) \pi(\Theta) \]

Better prospects.

**Example:** Posterior mean of \( \Theta \), estimated by

\[ \frac{1}{N-b} \sum_{t=b+1}^{n} \Theta^{(t)} \]
Posterior Variance by

\[
\frac{1}{N-5-1} \sum_{t=b+1}^{N} (\theta_t - \bar{\theta}_t)^2
\]

where

\[
\bar{\theta}_t = \frac{1}{N-b} \sum_{t=b+1}^{N} \theta_t
\]