MA30118 - Solution Sheet Five

Simon Shaw
s.c.shaw@maths.bath.ac.uk

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1. **The values of Bath building contracts (in millions of pounds) for a 12-month period are (in time order)** 17, 21, 19, 23, 18, 16, 20, 18, 22, 20, 15, 22

   (a) **Use $\alpha = 0.1$ to compute the exponential smoothing values for the time series using the month 1 figure as the initial forecast. Calculate the forecast error for each forecast.**

Month, t	X_t	$\hat{X}_t = M_{t-1} = \alpha X_{t-1} + (1-\alpha)M_{t-2}$	$X_t - \hat{X}_t$
1	17		
2	21	$M_1 = 17$	$21 - 17 = 4$
3	19	$M_2 = 0.1(21) + 0.9(17) = 17.4$	$19 - 17.4 = 1.6$
4	23	$M_3 = 0.1(19) + 0.9(17.4) = 17.56$	$23 - 17.56 = 5.44$
5	18	$M_4 = 0.1(23) + 0.9(17.56) = 18.104$	$18 - 18.104 = -0.104$
6	16	$M_5 = 0.1(18) + 0.9(18.104) = 18.0936$	$16 - 18.0936 = -2.0936$
7	20	$M_6 = 17.88424$	2.11576
8	18	$M_7 = 18.095816$	-0.095816
9	22	$M_8 = 18.0862344$	3.9137656
10	20	$M_9 = 18.47761096$	1.52238904
11	15	$M_{10} = 18.62984986$	-3.62984986
12	22	$M_{11} = 18.26686488$	3.73313512

   (b) **What is the forecast for next month? And the month following that?**
   The forecast for all future months (so next month is $n = 1$, et cetera) is given by

$$\hat{X}_{12+n} = M_{12} = 0.1(22) + 0.9(18.26686488) = 18.64017839.$$

   (c) **What is the value of the mean square deviation (MSD)?**

$$\begin{aligned} MSD &= \{4^2 + 1.6^2 + \cdots + (-3.62984986)^2 + 3.73313512^2\}/11 \\ &= 101.7805354/11 \\ &= 9.252775949. \end{aligned}$$

2. **Let X_t denote the quarterly earning per share of a corporation in quarter t. Using a season of four, a Holt-Winters multiplicative model with smoothing parameters $\alpha = 0.5$, $\gamma = 0.6$ and $\delta = 0.7$ was fitted. The data is summarised**

below.

t	X_t	M_t	T_t	S_t
1	0.712	0.5287729	0.01678876	1.3616536
2	0.584	0.6020379	0.05067450	0.9450805
3	0.620	0.6900502	0.07307715	0.8846493
4	0.620	0.7430797	0.06104857	0.8413058
5	0.891	0.7292399	0.01611553	1.2637702
6	0.570	0.6742393	−0.02655412	0.8753022
7	0.540			
8	0.690			

(a) **Find M_7, T_7 and S_7.**

Using the smoothing equations given in lectures with a season of $s = 4$ we have

$$\begin{aligned}
M_7 &= \alpha \frac{X_7}{S_3} + (1-\alpha)(M_6 + T_6) \\
&= 0.5\left(\frac{0.540}{0.8846493}\right) + 0.5(0.6742393 - 0.02655412) \\
&= 0.62904828, \\
T_7 &= \gamma(M_7 - M_6) + (1-\gamma)T_6 \\
&= 0.6(0.62904828 - 0.6742393) + 0.4(-0.02655412) \\
&= -0.03773626, \\
S_7 &= \delta \frac{X_7}{M_7} + (1-\delta)S_3 \\
&= 0.7\left(\frac{0.540}{0.62904828}\right) + 0.3(0.8846493) \\
&= 0.866302561.
\end{aligned}$$

(b) **Find M_8, T_8 and S_8.**

In a similar vein to the previous part, we have

$$\begin{aligned}
M_8 &= \alpha \frac{X_8}{S_4} + (1-\alpha)(M_7 + T_7) \\
&= 0.5\left(\frac{0.690}{0.8413058}\right) + 0.5(0.62904828 - 0.03773626) \\
&= 0.705732821, \\
T_8 &= \gamma(M_8 - M_7) + (1-\gamma)T_7 \\
&= 0.6(0.705732821 - 0.62904828) + 0.4(-0.03773626) \\
&= 0.03091622, \\
S_8 &= \delta \frac{X_8}{M_8} + (1-\delta)S_4 \\
&= 0.7\left(\frac{0.690}{0.705732821}\right) + 0.3(0.8413058) \\
&= 0.93678672.
\end{aligned}$$

(c) **Find the forecast of the quarterly earning per share in quarters 9, 10 and 15.**

The n-step ahead forecast from X_8, with $s = 4$, is given by:

$$\hat{X}_{8+n} = \begin{cases} (M_8 + nT_8)S_{8+n-4} & \text{for } n = 1, 2, 3, 4 \\ (M_8 + nT_8)S_{8+n-8} & \text{for } n = 5, 6, 7, 8 \end{cases} \quad \text{and so on,}$$

so that

$$\begin{aligned}
\hat{X}_9 &= (M_8 + T_8)S_5 \\
&= (0.705732821 + 0.03091622)(1.2637702) \\
&= 0.930955105, \\
\hat{X}_{10} &= (M_8 + 2T_8)S_6 \\
&= \{0.705732821 + 2(0.03091622)\}(0.8753022) \\
&= 0.671851561, \\
\hat{X}_{15} &= (M_8 + 7T_8)S_7 \\
&= \{0.705732821 + 7(0.03091622)\}(0.866302561) \\
&= 0.798857754.
\end{aligned}$$

3. **Let β_0 and β_1 be constants, $s_t = s_{t-12}$ for all t, and $\{I_t\}$ a sequence of independent random quantities with zero mean and constant variance.**

(a) **If $X_t = (\beta_0 + \beta_1 t)s_t + I_t$, show that $\nabla_{12}^2 X_t$ is a stationary process.**

$$\begin{aligned}
\nabla_{12} X_t &= X_t - X_{t-12} \\
&= (\beta_0 + \beta_1 t)s_t + I_t - [\{\beta_0 + \beta_1(t-12)\}s_{t-12} + I_{t-12}] \\
&= (\beta_0 + \beta_1 t)(s_t - s_{t-12}) + 12\beta_1 s_{t-12} + I_t - I_{t-12} \\
&= 12\beta_1 s_{t-12} + I_t - I_{t-12},
\end{aligned}$$

since $s_t = s_{t-12}$ for all t. Thus,

$$\begin{aligned}
\nabla_{12}^2 X_t &= \nabla_{12}(\nabla_{12} X_t) \\
&= \nabla_{12}(12\beta_1 s_{t-12} + I_t - I_{t-12}) \\
&= (12\beta_1 s_{t-12} + I_t - I_{t-12}) - (12\beta_1 s_{t-24} + I_{t-12} - I_{t-24}) \\
&= 12\beta_1(s_{t-12} - s_{t-24}) + I_t - 2I_{t-12} + I_{t-24} \\
&= I_t - 2I_{t-12} + I_{t-24},
\end{aligned}$$

since $s_{t-12} = s_{t-24}$. Hence, $\nabla_{12}^2 X_t$ only involves the sum of independent random quantities with zero mean and constant variance and is thus stationary.

(b) **If $X_t = \beta_0 + \beta_1 t + s_t + I_t$, show that $\nabla\nabla_{12} X_t$ is a stationary process.**

$$\begin{aligned}
\nabla_{12} X_t &= X_t - X_{t-12} \\
&= (\beta_0 + \beta_1 t + s_t + I_t) - \{\beta_0 + \beta_1(t-12) + s_{t-12} + I_{t-12}\} \\
&= 12\beta_1 + I_t - I_{t-12}.
\end{aligned}$$

We note that $\nabla_{12}X_t$ is thus a stationary series. We now check that if we difference a stationary series, the result is still a stationary series.

$$\begin{aligned}
\nabla\nabla_{12}X_t &= \nabla(\nabla_{12}X_t) \\
&= \nabla(12\beta_1 + I_t - I_{t-12}) \\
&= (12\beta_1 + I_t - I_{t-12}) - (12\beta_1 + I_{t-1} - I_{t-13}) \\
&= I_t - I_{t-1} - I_{t-12} + I_{t-13}.
\end{aligned}$$

Hence, $\nabla\nabla_{12}X_t$ only involves the sum of independent random quantities with zero mean and constant variance and is thus stationary.

4. **Forecasting from a fitted Box-Jenkins model. As mentioned in class such forecasts are obtained as follows. Suppose we have fitted the $ARMA(2,1)$ model:**

$$X_s = \phi_1 X_{s-1} + \phi_2 X_{s-2} + a_s + \theta_1 a_{s-1},$$

and that the latest available value is X_t. We wish to forecast ahead at $s = t+1, t+2, \ldots$.

Write down the forecasts $\hat{X}_{t+1}$, $\hat{X}_{t+2}$ and $\hat{X}_{t+3}$ using the following rules:

(a) **Use the above equation with the estimates $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\theta}_1$ and**

(b) **if the equation involves a_s where $s \leq t$ use the residual from the fit, $\hat{a}_s = X_s - \hat{X}_s$, otherwise if $s > t$, use $a_s = 0$, its expected value,**

(c) **if the equation involves X_s where $s \leq t$ use the actual recorded value of X_t but if $s > t$ use the forecast $\hat{X}_s$.**

$$X_{t+1} = \phi_1 X_t + \phi_2 X_{t-1} + a_{t+1} + \theta_1 a_t.$$

Now, X_t and X_{t-1} are available but X_{t+1} is replaced by the forecast value $\hat{X}_{t+1}$. ϕ_1, ϕ_2, θ_1 are replaced by their estimates $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\theta}_1$. a_{t+1} is replaced by its expected value 0 and a_t by the residual from the fit, $\hat{a}_t$. Thus,

$$\hat{X}_{t+1} = \hat{\phi}_1 X_t + \hat{\phi}_2 X_{t-1} + \hat{\theta}_1 \hat{a}_t.$$

Similarly,

$$X_{t+2} = \phi_1 X_{t+1} + \phi_2 X_t + a_{t+2} + \theta_1 a_{t+1}.$$

Now, X_t is available but X_{t+2}, X_{t+1} are replaced by the forecast values $\hat{X}_{t+2}$, $\hat{X}_{t+1}$ respectively. ϕ_1, ϕ_2, θ_1 are replaced by their estimates $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\theta}_1$. a_{t+2}, a_{t+1} are replaced by their expected values: 0. Thus,

$$\hat{X}_{t+2} = \hat{\phi}_1 \hat{X}_{t+1} + \hat{\phi}_2 X_t.$$

Finally, for X_{t+3} we have

$$X_{t+3} = \phi_1 X_{t+2} + \phi_2 X_{t+1} + a_{t+3} + \theta_1 a_{t+2}.$$

Now, X_{t+3}, X_{t+2}, X_{t+1} are replaced by the forecast values $\hat{X}_{t+3}$, $\hat{X}_{t+2}$, $\hat{X}_{t+1}$ respectively. ϕ_1, ϕ_2, θ_1 are replaced by their estimates $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\theta}_1$. a_{t+3}, a_{t+2} are replaced by their expected values: 0. Thus,

$$\hat{X}_{t+3} = \hat{\phi}_1 \hat{X}_{t+2} + \hat{\phi}_2 \hat{X}_{t+1}.$$

5. **The $ARIMA(1,1,1)$ model, which has first-order differencing, can be written in the form**

$$W_t = \gamma + \phi_1 W_{t-1} + a_t - \theta_1 a_{t-1}$$

where $W_t = \nabla X_t$.

(a) **Express the model in terms of $\{X_t\}$ and $\{a_t\}$ rather than $\{W_t\}$ and $\{a_t\}$.**
We replace W_t by $X_t - X_{t-1}$ and W_{t-1} by $X_{t-1} - X_{t-2}$ so that:

$$\begin{aligned} X_t - X_{t-1} &= \gamma + \phi_1(X_{t-1} - X_{t-2}) + a_t - \theta_1 a_{t-1} \Leftrightarrow \\ X_t &= \gamma + (1+\phi_1)X_{t-1} - \phi_1 X_{t-2} + a_t - \theta_1 a_{t-1}. \end{aligned}$$

(b) **Use this to derive the forecasts for one, two and three steps ahead from time t using the usual rules.**

$$X_{t+1} = \gamma + (1+\phi_1)X_t - \phi_1 X_{t-1} + a_{t+1} - \theta_1 a_t.$$

Now, X_t and X_{t-1} are available but X_{t+1} is replaced by the forecast value $\hat{X}_{t+1}$. γ, ϕ_1, θ_1 are replaced by their estimates $\hat{\gamma}$, $\hat{\phi}_1$, $\hat{\theta}_1$. a_{t+1} is replaced by its expected value 0 and a_t by the residual from the fit, $\hat{a}_t$. Thus,

$$\hat{X}_{t+1} = \hat{\gamma} + (1+\hat{\phi}_1)X_t - \hat{\phi}_1 X_{t-1} - \hat{\theta}_1 \hat{a}_t.$$

Similarly,

$$X_{t+2} = \gamma + (1+\phi_1)X_{t+1} - \phi_1 X_t + a_{t+2} - \theta_1 a_{t+1}.$$

Now, X_t is available but X_{t+2}, X_{t+1} are replaced by the forecast values $\hat{X}_{t+2}$, $\hat{X}_{t+1}$ respectively. γ, ϕ_1, θ_1 are replaced by their estimates $\hat{\gamma}$, $\hat{\phi}_1$, $\hat{\theta}_1$. a_{t+2}, a_{t+1} are replaced by their expected values: 0. Thus,

$$\hat{X}_{t+2} = \hat{\gamma} + (1+\hat{\phi}_1)\hat{X}_{t+1} - \hat{\phi}_1 X_t.$$

Finally, for X_{t+3} we have

$$X_{t+3} = \gamma + (1+\phi_1)X_{t+2} - \phi_1 X_{t+1} + a_{t+3} - \theta_1 a_{t+2}.$$

Now, X_{t+3}, X_{t+2}, X_{t+1} are replaced by the forecast values $\hat{X}_{t+3}$, $\hat{X}_{t+2}$, $\hat{X}_{t+1}$ respectively. γ, ϕ_1, θ_1 are replaced by their estimates $\hat{\gamma}$, $\hat{\phi}_1$, $\hat{\theta}_1$. a_{t+3}, a_{t+2} are replaced by their expected values: 0. Thus,

$$\hat{X}_{t+3} = \hat{\gamma} + (1+\hat{\phi}_1)\hat{X}_{t+2} - \hat{\phi}_1 \hat{X}_{t+1}.$$