

MA30118 - Solution Sheet Four

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1. A double sampling scheme has $n_1 = 50$, $n_2 = 70$, $c_1 = 0$ and $c_2 = c_3 = 2$.

(a) If the $AQL = 0.01$ and $LTPD = 5$, calculate (to 1 decimal place) the producer's risk and the consumer's risk.

Let X_1 denote the number of defectives in the first sample and X_2 the number of defectives in the second sample. Then, from notes,

$$OC(p) = P(X_1 \leq c_1 | p) + \sum_{d_1=c_1+1}^{c_2} P(X_2 \leq c_3 - d_1 | p) P(X_1 = d_1 | p).$$

Substituting $c_1 = 0$, $c_2 = c_3 = 2$ we find

$$\begin{aligned} OC(p) &= P(X_1 = 0 | p) + \sum_{d_1=1}^2 P(X_2 \leq 2 - d_1 | p) P(X_1 = d_1 | p) \\ &= P(X_1 = 0 | p) + P(X_2 \leq 1 | p) P(X_1 = 1 | p) + \\ &\quad P(X_2 = 0 | p) P(X_1 = 2 | p) \tag{1} \\ &= (1-p)^{50} + \{(1-p)^{70} + 70p(1-p)^{69}\} 50p(1-p)^{49} + \\ &\quad (1-p)^{70} 1225p^2(1-p)^{48} \\ &= (1-p)^{50} + 50p(1-p)^{118}(1+69p) + 1225p^2(1-p)^{118}. \\ &= (1-p)^{50} + 25p(1-p)^{118}(2+187p^2). \tag{2} \end{aligned}$$

Let p_1 denote the AQL. Then $p_1 = 0.01$ and the producer's risk,

$$\alpha = 1 - OC(0.01) = 1 - 0.900537674 = 0.1. \text{ (1DP)}$$

Let $100p_2$ denote the LTPD. Then $p_2 = 0.05$ and the consumer's risk,

$$\beta = OC(0.05) = 0.110309988 = 0.1. \text{ (1DP)}$$

An alternative is to calculate the probabilities using (1). For $p = 0.01$, we find

$$\begin{aligned} P(X_1 = 0 | p = 0.01) &= 0.605006067 & P(X_2 = 0 | p = 0.01) &= 0.494838659 \\ P(X_1 = 1 | p = 0.01) &= 0.305558619 & P(X_2 = 1 | p = 0.01) &= 0.34988592 \\ P(X_1 = 2 | p = 0.01) &= 0.075618042 \end{aligned}$$

Hence,

$$\begin{aligned} OC(0.01) &= 0.605006067 + (0.494838659 + 0.34988592)0.305558619 + \\ &\quad 0.494838659(0.075618042) \\ &= 0.900532213. \end{aligned}$$

For $p = 0.05$, we find

$$\begin{aligned} P(X_1 = 0|p = 0.05) &= 0.076944975 & P(X_2 = 0|p = 0.05) &= 0.02758369 \\ P(X_1 = 1|p = 0.05) &= 0.202486777 & P(X_2 = 1|p = 0.05) &= 0.101624122 \\ P(X_1 = 2|p = 0.05) &= 0.26110137 \end{aligned}$$

Hence,

$$\begin{aligned} OC(0.01) &= 0.076944975 + (0.02758369 + 0.101624122)0.202486777 + \\ &\quad 0.02758369(0.26110137) \\ &= 0.110309987. \end{aligned}$$

- (b) **If a rectification scheme is applied, plot the AOQ (= $pOC(p)$) for the scheme and estimate the AOQL. [Either by using your favourite graphics package or by evaluating AOQ(p) at a number of values of p , say 0, 0.01, 0.02, 0.04, 0.06, 0.08, 0.10, 0.12, 0.14, 0.16. Restricting p to between 0 and 0.16 should give a nice plot.]**

Using equation (2), we have that

$$\begin{aligned} AOQ(p) &= pOC(p) \\ &= p(1-p)^{50} + 25p^2(1-p)^{118}(2 + 187p^2). \end{aligned}$$

Figure 1 gives a plot of the AOQ. Note that the AOQ is virtually zero for $p > 0.16$. As commented upon in lectures, the AOQ has a maximum between 0 and 1. This maximum occurs at about $p = 0.02118$, the value being approximately 0.0126. This is the AOQL, the average outgoing quality limit. The worse case scenario for the consumer is that he will receive approximately 1.26% of defective items. If you are plotting the AOQ at a number of points, you should find that:

p	$AOQ(p)$
0	0
0.01	0.009005377
0.02	0.012575016
0.04	0.008263562
0.06	0.003522557
0.08	0.001382032
0.10	0.000536009
0.12	0.000203529
0.14	0.000074568
0.16	0.000026211

- (c) **Plot the ASN for the scheme and estimate the maximum value of the ASN. [Again either by hand or by your favourite graphics package; restricting p to between 0 and 0.2 should give a nice plot.]**

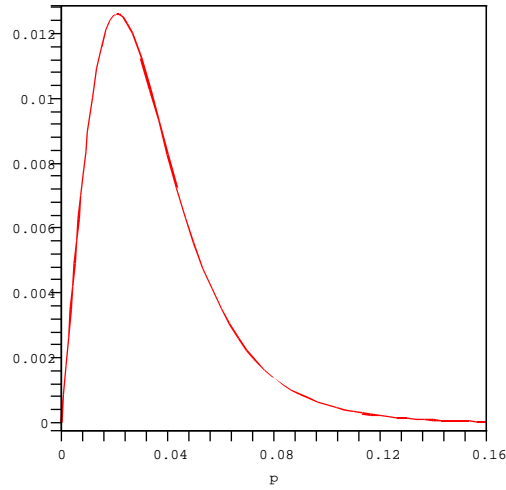


Figure 1: The AOQ, $AOQ(p) = p(1-p)^{50} + 25p^2(1-p)^{118}(2 + 187p^2)$ for Question 2(b).

$$\begin{aligned} ASN(p) &= 50 + 70P(1 \leq X_1 \leq 2) \\ &= 50 + 70(50p(1-p)^{49} + 1225p^2(1-p)^{48}). \end{aligned}$$

Figure 2 gives a plot of the ASN. Note that $ASN(p)$ is virtually fifty for $p > 0.2$. If you are plotting the ASN at a number of points, you should find that:

p	$ASN(p)$
0	50
0.01	76.68236634
0.02	89.01818001
0.04	88.27797474
0.06	75.96410221
0.08	64.73554597
0.10	57.46024868
0.12	53.47121023
0.14	51.50861712
0.16	50.61830208
0.18	50.24038488
0.20	50.08897997

- (d) **Compare your ASN with the sample size required for the corresponding single sampling scheme having the same AQL, LTPD, producer's risk and consumer's risk. Comment on your results.**

We take $p_1 = 0.01$, $p_2 = 0.05$, $\alpha = 0.1$ and $\beta = 0.1$.

- i. Calculate p_2/p_1 . In this case, $p_2/p_1 = 0.05/0.01 = 5$.
- ii. Use Table 6 with $\alpha = 0.1$ and $\beta = 0.1$ to find the smallest c such that

$$r(c-1) > \frac{p_2}{p_1} > r(c).$$

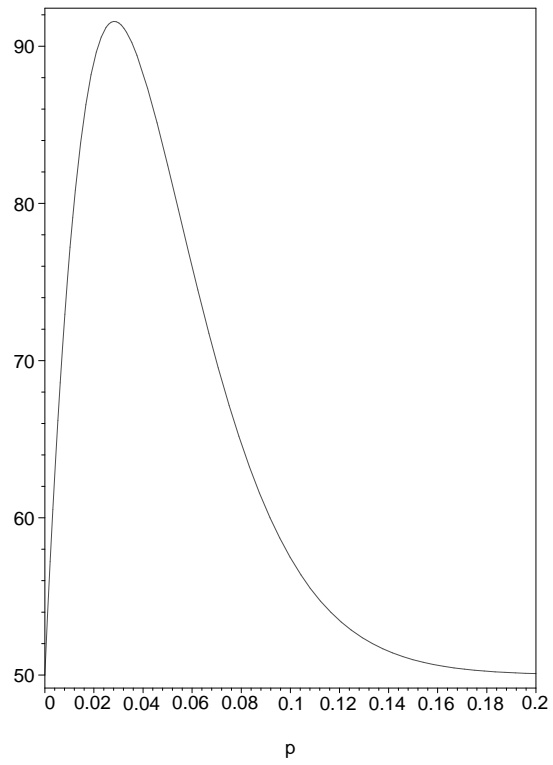


Figure 2: The ASN, $ASN(p) = 50 + 70(50p(1 - p)^{49} + 1225p^2(1 - p)^{48})$ for Question 3.

We use the first column of Table 6(a). We see that $r(1) = 7.31$ and $r(2) = 4.83$ so that $7.31 > 5 > 4.83$. We thus choose $c = 2$.

iii. The possible values of n are

$$\frac{\chi_{1-\beta, 2(c+1)}^2}{2p_2} \leq n \leq \frac{\chi_{\alpha, 2(c+1)}^2}{2p_1},$$

where Table 3 is used to find $\chi_{1-\beta, 2(c+1)}^2$ and $\chi_{\alpha, 2(c+1)}^2$. In our case, $2(c+1) = 2 \times 3 = 6$. From Table 3 we find $\chi_{0.9, 6}^2 = 10.64$ and $\chi_{0.1, 6}^2 = 2.20$. Thus, our possible values of n are

$$\begin{aligned} \frac{10.64}{2(0.05)} &\leq n \leq \frac{2.20}{2(0.01)} \Rightarrow \\ 106.4 &\leq n \leq 110. \end{aligned}$$

Whatever the value of p , we see, from Figure 2, that $ASN(p) < 100$ so that the double sample scheme is more efficient than the “equivalent” single sample scheme in this example, in the sense that the expected sample size is always smaller than the sample size in the single sample scheme.

2. **A certain protective material is satisfactory if it is perforated by not more than 15% of pellets fired under standard conditions. It is not acceptable if the percentage is more than 30. The consumer’s risk is 0.02 and the producer’s risk is 0.01.**

(a) **Construct a sequential probability ratio test to determine whether the material should be accepted.**

We calculate:

$$\begin{aligned} k &= \log \left\{ \frac{0.3 \times 0.85}{0.15 \times 0.70} \right\} = 0.8873031952; \\ a &= \frac{1}{k} \log \left\{ \frac{0.02}{0.99} \right\} = \frac{-3.901972670}{0.8873031952} = -4.397564092; \\ b &= \frac{1}{k} \log \left\{ \frac{0.98}{0.01} \right\} = \frac{4.584967479}{0.8873031952} = 5.167306400; \\ c &= \frac{1}{k} \log \left\{ \frac{0.85}{0.70} \right\} = \frac{0.194156014}{0.8873031952} = 0.2188158628. \end{aligned}$$

Let d be the number of defectives in the first n items. Our sequential sampling rule is:

- i. Acceptance region: if $d \leq -4.397564092 + 0.2188158628n$ then accept the batch.
- ii. Rejection region: if $d \geq 5.167306400 + 0.2188158628n$ then reject the batch.
- iii. Continuation rule: if

$$-4.397564092 + 0.2188158628n < d < 5.167306400 + 0.2188158628n$$

then continue sampling (until the batch is either accepted or rejected).

- (b) **What is the smallest number of observations at which the rejection of the batch can occur?**

The first n at which rejection can occur is $\min\{n : b + cn \leq n\}$, that is the smallest integer

$$n \geq \frac{b}{1-c} = \frac{5.167306400}{1-0.2188158628} = 6.614709841.$$

So, the smallest number of observations at which rejection can occur is 7.

- (c) **What is the smallest number of observations at which the acceptance of the batch can occur?**

The first n at which acceptance can occur is given by $\min\{n : a + cn \geq 0\}$, that is the smallest integer

$$n \geq \frac{-a}{c} = \frac{4.397564092}{0.2188158628} = 20.09709916.$$

So, the smallest number of observations at which acceptance can occur is 21.

- (d) **Carry out the test on the following data: perforations occurred at (and only at) shots numbered 8, 15, 20, 32, 37, 40, 46, 54, 61, 67, 77. [Of course, in reality, no shots are fired once a decision has been made.]**

The step-function showing the number of observed defectives against the total number of samples is shown in Figure 3. We continue sampling until the step-function crosses either the upper and lower boundary upon which we cease sampling. We observe that we first cross a boundary, the lower boundary, after we observed seven defective items but before we observe eight defective items. We now find the exact sample number when the step-function crosses the lower boundary by finding the first integer n such that $a + cn \geq 7$. That is,

$$n \geq \frac{7-a}{c} = \frac{7+4.397564092}{0.2188158628} = 52.08746726.$$

We stop and accept the batch after the 53rd sample.

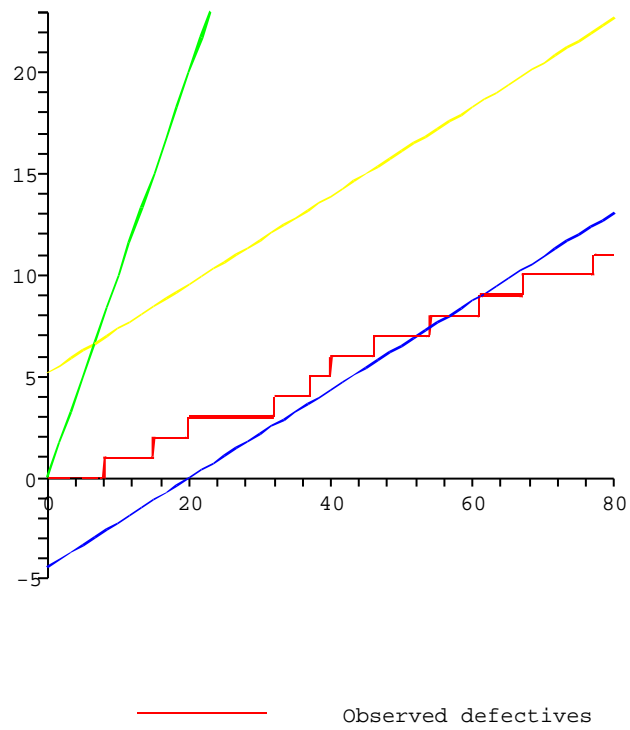


Figure 3: Step-function and sampling boundaries for the SPRT.