MA30118 - Solution Sheet Three

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1. **Plot the OC curve for a single sampling scheme with $n = 50$ and $c = 2$. Taking the producer's risk and consumer's risk as 0.05 and 0.1 respectively, use the graph to estimate the AQL and LTPD. [You may use your favourite graphical package to achieve this, or do it by plotting the OC at a number of values of p, say 0, 0.02, 0.04, 0.06, 0.08, 0.10, 0.12 and 1.]**

   Let X denote the number of defective items in the batch. Then

$$\begin{aligned} OC(p) &= P(X \leq 2 \mid p) = \sum_{d=0}^{2} \binom{50}{d} p^d (1-p)^{50-d} \\ &= (1-p)^{50} + 50p(1-p)^{49} + 1225p^2(1-p)^{48} \\ &= (1-p)^{48}(1 + 48p + 1176p^2). \end{aligned}$$

   Figure 1 gives a plot of the OC curve. Note that $OC(p)$ is virtually zero for $p > 0.2$. If you are plotting the OC at a number of points, you should find that:

p	$OC(p)$
0	1
0.02	0.921572251
0.04	0.676714004
0.06	0.416246472
0.08	0.225974275
0.10	0.111728756
0.12	0.051264175
1	0

   The producer's risk is 0.05. Let p_1 denote the AQL. Then p_1 is such that

$$OC(p_1) = (1-p_1)^{48}(1 + 48p_1 + 1176p_1^2) = 0.95.$$

   From Figure 1 we see that, approximately, $p_1 = 0.0165$. The consumer's risk is 0.1. Let $100p_2$ denote the LTPD. Then p_2 is such that

$$OC(p_2) = (1-p_2)^{48}(1 + 48p_2 + 1176p_2^2) = 0.1.$$

   From Figure 1 we see that, approximately, $p_2 = 0.103$. Hence, $LTPD = 10.3$.

2. **Suppose that we wish to construct a single sample scheme with $AQL = 0.02$, $LTPD = 6$, and producer's and consumer's risk equal to 0.05 and 0.1 respectively.**

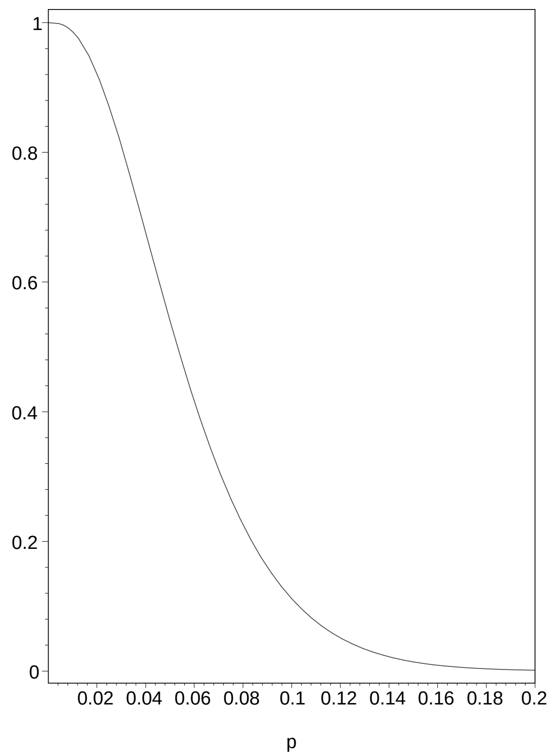

Figure 1: The OC curve, $OC(p) = (1-p)^{48}(1 + 48p + 1176p^2)$ for Question 1.

(a) **Use tables to determine suitable values of n and c.**

We take $p_1 = 0.02$, $p_2 = 0.06$, $\alpha = 0.05$ and $\beta = 0.1$.

i. Calculate p_2/p_1. In this case, $p_2/p_1 = 0.06/0.02 = 3$.

ii. Use Table 6 with $\alpha = 0.05$ and $\beta = 0.1$ to find the smallest c such that

$$r(c-1) > \frac{p_2}{p_1} > r(c).$$

We use the first column of Table 6(b). We see that $r(6) = 3.21$ and $r(7) = 2.96$ so that $3.21 > 3 > 2.96$. We thus choose $c = 7$.

iii. The possible values of n are

$$\frac{\chi^2_{1-\beta,2(c+1)}}{2p_2} \leq n \leq \frac{\chi^2_{\alpha,2(c+1)}}{2p_1},$$

where Table 3 is used to find $\chi^2_{1-\beta,2(c+1)}$ and $\chi^2_{\alpha,2(c+1)}$. In our case, $2(c+1) = 2 \times 8 = 16$. From Table 3 we find $\chi^2_{0.9,16} = 23.54$ and $\chi^2_{0.05,16} = 7.96$. Thus, our possible values of n are

$$\begin{array}{ccc} \frac{23.54}{2(0.06)} & \leq n \leq & \frac{7.96}{2(0.02)} \Rightarrow \\ 196.167 & \leq n \leq & 199. \end{array}$$

(b) **Check your answers by calculating the producer's risk and the consumer's risk for your choice of n and c with the given AQL and $LTPD$.**

Taking $c = 7$, the operating characteristic is

$$OC(p) \;=\; P(X \leq 7 \mid p) \;=\; \sum_{d=0}^{7} \binom{n}{d} p^d (1-p)^{n-d},$$

where X is the number of defective items in the sample. Possible sample sizes are 197, 198, and 199. The producer's risk is $1 - OC(0.02)$ while the consumer's risk is $OC(0.06)$. We find

n	Producer's risk	Consumer's risk
197	0.0458899495	0.09078185050
198	0.0470203031	0.08808086537
199	0.0481686481	0.08544888970

We note that the producer's risk is always less than 0.05 and the consumer's risk is always less than 0.1 for each of the choices of n and the specified c.

3. **A double sampling scheme has $n_1 = 50$, $n_2 = 70$, $c_1 = 1$ and $c_2 = c_3 = 3$.**

(a) **If the $AQL = 0.01$ and $LTPD = 10$, calculate the producer's risk and the consumer's risk.**

Let X_1 denote the number of defectives in the first sample and X_2 the number of defectives in the second sample. Then, $X_1 \sim Bin(50, p)$, $X_2 \sim Bin(70, p)$ and, from notes,

$$OC(p) = P(X_1 \leq c_1 \mid p) + \sum_{d_1=c_1+1}^{c_2} P(X_1 = d_1 \mid p) P(X_2 \leq c_3 - d_1 \mid p).$$

Substituting $c_1 = 1$, $c_2 = c_3 = 3$ we find

$$\begin{aligned} OC(p) &= P(X_1 \leq 1 \mid p) + \sum_{d_1=2}^{3} P(X_1 = d_1 \mid p) P(X_2 \leq 3 - d_1 \mid p) \\ &= P(X_1 \leq 1 \mid p) + P(X_1 = 2 \mid p) P(X_2 \leq 1 \mid p) + \\ &\quad P(X_1 = 3 \mid p) P(X_2 = 0 \mid p) \qquad (1) \\ &= \{(1-p)^{50} + 50p(1-p)^{49}\} + 1225p^2(1-p)^{48}\{(1-p)^{70} + \\ &\quad 70p(1-p)^{69}\} + 19600p^3(1-p)^{47}(1-p)^{70}. \end{aligned}$$

Let p_1 denote the AQL. Then $p_1 = 0.01$ and the producer's risk,

$$\alpha = 1 - OC(0.01) = 1 - 0.9804885775 = 0.0195114225.$$

Let $100p_2$ denote the LTPD. Then $p_2 = 0.1$ and the consumer's risk,

$$\beta = OC(0.1) = 0.03430136519.$$

An alternative is to calculate the probabilities using (1). For $p = 0.01$, we find

$$\begin{aligned} P(X_1 = 0 \mid p = 0.01) &= 0.6050060671, \\ P(X_1 = 1 \mid p = 0.01) &= 0.3055586198, \\ P(X_1 = 2 \mid p = 0.01) &= 0.07561804226, \\ P(X_1 = 3 \mid p = 0.01) &= 0.01222109774, \\ P(X_2 = 0 \mid p = 0.01) &= 0.4948386596, \\ P(X_2 = 1 \mid p = 0.01) &= 0.3498859209. \end{aligned}$$

Hence,

$$\begin{aligned} OC(0.01) &= (0.6050060671 + 0.3055586198) + \\ &\quad 0.07561804226(0.4948386596 + 0.3498859209) + \\ &\quad 0.01222109774(0.4948386596) \\ &= 0.9804885775 \end{aligned}$$

so that the producer's risk is $1 - 0.9804885775 = 0.0195114225$. For $p = 0.1$, we find

$$\begin{aligned} P(X_1 = 0 \mid p = 0.1) &= 0.005153775207, \\ P(X_1 = 1 \mid p = 0.1) &= 0.02863208448, \\ P(X_1 = 2 \mid p = 0.1) &= 0.07794289665, \\ P(X_1 = 3 \mid p = 0.1) &= 0.1385651496, \\ P(X_2 = 0 \mid p = 0.1) &= 0.0006265787482, \\ P(X_2 = 1 \mid p = 0.1) &= 0.004873390264. \end{aligned}$$

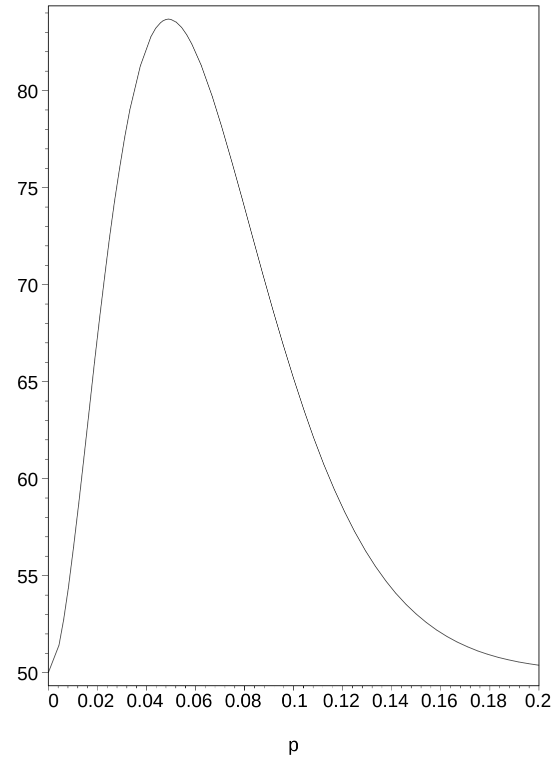

Figure 2: The ASN, $ASN(p) = 50 + 70\{1225p^2(1-p)^{48} + 19600p^3(1-p)^{47}\}$ for Question 3.

Hence,

$$\begin{aligned} OC(0.1) &= (0.005153775207 + 0.02863208448) + \\ &\quad 0.07794289665(0.0006265787482 + 0.004873390264) + \\ &\quad 0.1385651496(0.0006265787482) \\ &= 0.03430136519 \end{aligned}$$

which is the consumer's risk.

(b) **Plot the ASN for the scheme. [Again, by either using your favourite graphics package or evaluating ASN(p) at a number of values of p, say 0, 0.02, 0.04, 0.06, 0.08, 0.10, 0.12, 0.14, 0.16, 0.18, 0.20 and 1.]**

$$\begin{aligned} ASN(p) &= 50 + 70P(2 \leq X_1 \leq 3 \mid p) \\ &= 50 + 70\{1225p^2(1-p)^{48} + 19600p^3(1-p)^{47}\}. \end{aligned}$$

Figure 2 gives a plot of the ASN. Note that $ASN(p)$ is virtually fifty for $p > 0.2$. If you are plotting the ASN at a number of points, you should find that:

p	$ASN(p)$
0	50
0.02	67.25293674
0.04	82.22716087
0.06	82.01100798
0.08	73.98085531
0.10	65.15556324
0.12	58.50041723
0.14	54.34789555
0.16	52.06100695
0.18	50.91460153
0.20	50.38245778
1	50