MA30118 - Solution Sheet Two

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1. **For a fee of £400, the manager in question 2. of Question Sheet One may call in a consultant. Let I_1 represent the event that the consultant predicts that business will decline, I_2 that the consultant predicts business will remain the same, I_3 that the consultant predicts business will increase moderately, and I_4 that the consultant predicts business will increase rapidly. Table 1 lists the conditional probabilities for predictions made by the consultant.**

	S_1	S_2	S_3	S_4
I_1	0.80	0.10	0.20	0.10
I_2	0.10	0.70	0.20	0.20
I_3	0.05	0.10	0.50	0.30
I_4	0.05	0.10	0.10	0.40

Table 1: Conditional probabilities for the consultant's predictions.

(a) **Find the $EVSI$.**

Firstly we find the posterior probabilities. These are given in Table 2 We now calculate the expected monetary value of each action given the prediction of the consultant, that is $EMV(A_i|I_k)$. For I_1 we find:

$$\begin{aligned}
EMV(A_1|I_1) &= 1451\left(\frac{12}{23}\right) + 1840\left(\frac{5}{46}\right) + 2050\left(\frac{5}{23}\right) + 2300\left(\frac{7}{46}\right) \\
&= 1752\tfrac{16}{23}; \\
EMV(A_2|I_1) &= -1091\left(\frac{12}{23}\right) + 1685\left(\frac{5}{46}\right) + 2430\left(\frac{5}{23}\right) + 2900\left(\frac{7}{46}\right) \\
&= 583\tfrac{1}{2}; \\
EMV(A_3|I_1) &= -2015\left(\frac{12}{23}\right) + 1100\left(\frac{5}{46}\right) + 3060\left(\frac{5}{23}\right) + 3561\left(\frac{7}{46}\right) \\
&= 275\tfrac{17}{46}; \\
EMV(A_4|I_1) &= -3460\left(\frac{12}{23}\right) - 1350\left(\frac{5}{46}\right) + 3340\left(\frac{5}{23}\right) + 4300\left(\frac{7}{46}\right) \\
&= -571\tfrac{12}{23}.
\end{aligned}$$

Thus, $EMV(I_1) = \max_i EMV(A_i|I_1) = 1752\frac{16}{23}$ under action A_1: we lay off two

The table for I_1:

j	S_j	$P(S_j)$	$P(I_1\|S_j)$	$P(I_1 \cap S_j)$		$P(S_j\|I_1)$	
1	S_1	0.15	0.8	$0.15 \times 0.8 =$	0.12	$0.12 \div 0.23 =$	$\frac{12}{23}$
2	S_2	0.25	0.1	$0.25 \times 0.1 =$	0.025	$0.025 \div 0.23 =$	$\frac{5}{46}$
3	S_3	0.25	0.2	$0.25 \times 0.2 =$	0.05	$0.05 \div 0.23 =$	$\frac{5}{23}$
4	S_4	0.35	0.1	$0.35 \times 0.1 =$	0.035	$0.035 \div 0.23 =$	$\frac{7}{46}$
		1		$P(I_1) =$	0.23		1

The table for I_2:

j	S_j	$P(S_j)$	$P(I_2\|S_j)$	$P(I_2 \cap S_j)$		$P(S_j\|I_2)$	
1	S_1	0.15	0.1	$0.15 \times 0.1 =$	0.015	$0.015 \div 0.31 =$	$\frac{3}{62}$
2	S_2	0.25	0.7	$0.25 \times 0.7 =$	0.175	$0.175 \div 0.31 =$	$\frac{35}{62}$
3	S_3	0.25	0.2	$0.25 \times 0.2 =$	0.05	$0.05 \div 0.31 =$	$\frac{5}{31}$
4	S_4	0.35	0.2	$0.35 \times 0.2 =$	0.07	$0.07 \div 0.31 =$	$\frac{7}{31}$
		1		$P(I_2) =$	0.31		1

The table for I_3:

j	S_j	$P(S_j)$	$P(I_3\|S_j)$	$P(I_3 \cap S_j)$		$P(S_j\|I_3)$	
1	S_1	0.15	0.05	$0.15 \times 0.05 =$	0.0075	$0.0075 \div 0.2625 =$	$\frac{1}{35}$
2	S_2	0.25	0.10	$0.25 \times 0.10 =$	0.025	$0.025 \div 0.2625 =$	$\frac{2}{21}$
3	S_3	0.25	0.50	$0.25 \times 0.50 =$	0.125	$0.125 \div 0.2625 =$	$\frac{10}{21}$
4	S_4	0.35	0.30	$0.35 \times 0.30 =$	0.105	$0.105 \div 0.2625 =$	$\frac{2}{5}$
		1		$P(I_3) =$	0.2625		1

The table for I_4:

j	S_j	$P(S_j)$	$P(I_4\|S_j)$	$P(I_4 \cap S_j)$		$P(S_j\|I_4)$	
1	S_1	0.15	0.05	$0.15 \times 0.05 =$	0.0075	$0.0075 \div 0.1975 =$	$\frac{3}{79}$
2	S_2	0.25	0.10	$0.25 \times 0.10 =$	0.025	$0.025 \div 0.1975 =$	$\frac{10}{79}$
3	S_3	0.25	0.10	$0.25 \times 0.10 =$	0.025	$0.025 \div 0.1975 =$	$\frac{10}{79}$
4	S_4	0.35	0.40	$0.35 \times 0.40 =$	0.14	$0.14 \div 0.1975 =$	$\frac{56}{79}$
		1		$P(I_4) =$	0.1975		1

Table 2: Calculating the posterior probabilities of the states of nature following the predictions of the consultant.

staff employees. For I_2 we find:

$$\begin{aligned}
EMV(A_1|I_2) &= 1451\left(\frac{3}{62}\right)+1840\left(\frac{35}{62}\right)+2050\left(\frac{5}{31}\right)+2300\left(\frac{7}{31}\right)\\
&= 1958\tfrac{57}{62};\\
EMV(A_2|I_2) &= -1091\left(\frac{3}{62}\right)+1685\left(\frac{35}{62}\right)+2430\left(\frac{5}{31}\right)+2900\left(\frac{7}{31}\right)\\
&= 1945\tfrac{6}{31};\\
EMV(A_3|I_2) &= -2015\left(\frac{3}{62}\right)+1100\left(\frac{35}{62}\right)+3060\left(\frac{5}{31}\right)+3561\left(\frac{7}{31}\right)\\
&= 1821\tfrac{7}{62};\\
EMV(A_4|I_2) &= -3460\left(\frac{3}{62}\right)-1350\left(\frac{35}{62}\right)+3340\left(\frac{5}{31}\right)+4300\left(\frac{7}{31}\right)\\
&= 580\tfrac{5}{31}.
\end{aligned}$$

Thus, $EMV(I_2) = \max_i EMV(A_i|I_2) = 1958\frac{57}{62}$ under action A_1: we lay off two staff employees. For I_3 we find:

$$\begin{aligned}
EMV(A_1|I_3) &= 1451\left(\frac{1}{35}\right)+1840\left(\frac{2}{21}\right)+2050\left(\frac{10}{21}\right)+2300\left(\frac{2}{5}\right)\\
&= 2112\tfrac{31}{35};\\
EMV(A_2|I_3) &= -1091\left(\frac{1}{35}\right)+1685\left(\frac{2}{21}\right)+2430\left(\frac{10}{21}\right)+2900\left(\frac{2}{5}\right)\\
&= 2446\tfrac{47}{105};\\
EMV(A_3|I_3) &= -2015\left(\frac{1}{35}\right)+1100\left(\frac{2}{21}\right)+3060\left(\frac{10}{21}\right)+3561\left(\frac{2}{5}\right)\\
&= 2928\tfrac{7}{105};\\
EMV(A_4|I_3) &= -3460\left(\frac{1}{35}\right)-1350\left(\frac{2}{21}\right)+3340\left(\frac{10}{21}\right)+4300\left(\frac{2}{5}\right)\\
&= 3083\tfrac{1}{21}.
\end{aligned}$$

Thus, $EMV(I_3) = \max_i EMV(A_i|I_3) = 3083\frac{1}{21}$ under action A_4: we increase the staff size by two employees. For I_4 we find:

$$\begin{aligned}
EMV(A_1|I_4) &= 1451\left(\frac{3}{79}\right)+1840\left(\frac{10}{79}\right)+2050\left(\frac{10}{79}\right)+2300\left(\frac{56}{79}\right)\\
&= 2177\tfrac{70}{79};\\
EMV(A_2|I_4) &= -1091\left(\frac{3}{79}\right)+1685\left(\frac{10}{79}\right)+2430\left(\frac{10}{79}\right)+2900\left(\frac{56}{79}\right)\\
&= 2535\tfrac{12}{79};\\
EMV(A_3|I_4) &= -2015\left(\frac{3}{79}\right)+1100\left(\frac{10}{79}\right)+3060\left(\frac{10}{79}\right)+3561\left(\frac{56}{79}\right)\\
&= 2974\tfrac{25}{79};\\
EMV(A_4|I_4) &= -3460\left(\frac{3}{79}\right)-1350\left(\frac{10}{79}\right)+3340\left(\frac{10}{79}\right)+4300\left(\frac{56}{79}\right)\\
&= 3168\tfrac{48}{79}.
\end{aligned}$$

Thus, $EMV(I_4) = \max_i EMV(A_i|I_4) = 3168\frac{48}{79}$ under action A_4: we increase the staff size by two employees.

The expected monetary value of the problem considering the consultant's information is

$$\begin{aligned} EMV &= EMV(I_1)P(I_1) + EMV(I_2)P(I_2) + EMV(I_3)P(I_3) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad +EMV(I_4)P(I_4) \\ &= 1752\tfrac{16}{23}(0.23) + 1958\tfrac{57}{62}(0.31) + 3083\tfrac{1}{21}(0.2625) + 3168\tfrac{48}{79}(0.1975) \\ &= 2445.485. \end{aligned}$$

The Bayes' decision rule for this problem is A_1 if I_1 or I_2 and A_4 if I_3 or I_4. The expected payoff with the sampling information is thus $EVwSI = 2445.485$. From question 2. of Question Sheet One, the expected payoff without the sampling information is $EVwoSI = 1995.15$. Hence,

$$EVSI \;=\; EVwSI - EVwoSI \;=\; 2445.485 - 1995.15 \;=\; 450.335.$$

(b) **What is the net efficiency?**

To calculate the net efficiency we must first find $EVPI$ using $EMVUC$. Note that $\pi^*(S_1) = 1451$, $\pi^*(S_2) = 1840$, $\pi^*(S_3) = 3340$ and $\pi^*(S_4) = 4300$. Thus,

$$EMVUC \;=\; 1451(0.15) + 1840(0.25) + 3340(0.25) + 4300(0.35) \;=\; 3017.65.$$

Hence,

$$EVPI \;=\; EMVUC - EVwoSI \;=\; 3017.65 - 1995.15 \;=\; 1022.5.$$

The consultant cost £400 so the net expected gain is

$$\text{Net expected gain} \;=\; EVSI - C \;=\; 450.335 - 400 \;=\; 50.335.$$

The net efficiency is thus

$$NE \;=\; \frac{\text{Net expected gain}}{EVPI} \times 100 \;=\; \frac{50.335}{1022.5} \times 100 \;=\; 4.9\%.$$

(c) **Comment on the value of the consultant.**

The benefit of the consultant is almost all absorbed in his fee. It may be better to look for alternative sources of information.

2. **I have been offered two investment opportunities, A and B, which require approximately the same cash outlay. The cash requirements mean that I can only afford to make at most one investment. I thus have three alternatives: make investment A (A_1); make investment B (A_2); or to not invest (A_3). The returns on my investments depends upon what happens to the stock market in the next year. With probability 0.3, the stock market may increase (S_1); with probability 0.5 it may remain stable (S_2) and with probability 0.2, the market may fall (S_3). The possible payoffs, in pounds as profits, are given in Table 3 below.**

	S_1	S_2	S_3
A_1	45,000	30,000	-75,000
A_2	75,000	-30,000	-45,000
A_3	0	0	0

Table 3: Payoffs for my investment choices in question 2.

(a) **Calculate the EMV of the investment decision problem and thus state the optimal decision under this criterion.**

We calculate the EMV for each action.

$$\begin{aligned} EMV(A_1) &= 45000(0.3) + 30000(0.5) - 75000(0.2) \ = \ 13500 \\ EMV(A_2) &= 75000(0.3) - 30000(0.5) - 45000(0.2) \ = \ -1500 \\ EMV(A_3) &= 0(0.3) + 0(0.5) + 0(0.2) \ = \ 0 \end{aligned}$$

Thus, $EMV = \max_i EMV(A_i) = 13500$ under action A_1: we make investment A.

(b) **Suppose that it is pointed out to me that both actions A_1 and A_2 could result in losses, so I decide to think about the risk of the investments. I elect to construct a utility function over my possible payoffs. I assign the following indifference probabilities.**

Profit	Indifference probability
£45,000	0.95
£30,000	0.90
£0	0.75
-£30,000	0.55
-£45,000	0.40

Construct the corresponding utility table and hence find the decision which maximises the expected utility. Comment on this decision.

Setting $U(£75,000) = 1$ and $U(-£75,000) = 0$ then the indifference probabilities correspond to our utilities. We may construct the utility table for the investment choice. This is Table 4.

	S_1	S_2	S_3
A_1	0.95	0.90	0
A_2	1	0.55	0.4
A_3	0.75	0.75	0.75

Table 4: Utilities for my investment choices.

We calculate the expected utility for each action.

$$\begin{aligned} EU(A_1) &= 0.95(0.3) + 0.90(0.5) + 0(0.2) \ = \ 0.735 \\ EU(A_2) &= 1(0.3) + 0.55(0.5) + 0.4(0.2) \ = \ 0.655 \\ EU(A_3) &= 0.75(0.3) + 0.75(0.5) + 0.75(0.2) \ = \ 0.75 \end{aligned}$$

Thus, $EU = \max_i EU(A_i) = 0.75$ under action A_3: we do not invest. Notice that when we consider utilities, our decision changes. We opt for the safe decision of not investing (our utility function is risk averse).

3. **A firm has three investment alternatives: A_1, A_2, and A_3. The return of these investments depends upon what happens to the stock market in the next year. With probability 0.4, the stock market may go up (S_1); with probability 0.3 it may remain stable (S_2) and with probability 0.3, the market may go down (S_3). The possible payoffs, in \$1000s as profits, are given in Table 5 below.**

	S_1	S_2	S_3
A_1	100	25	0
A_2	75	50	25
A_3	50	50	50

Table 5: Payoffs for my investment choices in question 3.

(a) **Calculate the EMV of the investment decision problem and thus state the optimal decision under this criterion.**

We calculate the EMV for each action.

$$\begin{aligned} EMV(A_1) &= 100(0.4) + 25(0.3) + 0(0.3) = 47.5 \quad (1) \\ EMV(A_2) &= 75(0.4) + 50(0.3) + 25(0.3) = 52.5 \quad (2) \\ EMV(A_3) &= 50(0.4) + 50(0.3) + 50(0.3) = 50 \quad (3) \end{aligned}$$

Thus, $EMV = \max_i EMV(A_i) = 52.5$ under action A_2.

(b) **For the lottery having a payoff of \$100,000 with probability p and \$0 with probability $1 - p$, two decision makers expressed the following indifference probabilities:**

	Indifference probability	
Profit	Decision Maker A	Decision Maker B
\$75,000	0.80	0.60
\$ 50,000	0.60	0.30
\$ 25,000	0.30	0.15

Find the most preferred decision for each decision maker using the expected utility approach.

For either decision maker, if we set $U(\$100,000) = 1$ and $U(\$0) = 0$ then the indifference probabilities correspond to their utilities. Firstly, we construct the utility table for decision maker A. This is Table 6. We calculate his expected utility for each action.

$$\begin{aligned} EU(A_1) &= 1(0.4) + 0.3(0.3) + 0(0.3) = 0.49 \quad (4) \\ EU(A_2) &= 0.8(0.4) + 0.6(0.3) + 0.3(0.3) = 0.59 \quad (5) \\ EU(A_3) &= 0.6(0.4) + 0.6(0.3) + 0.6(0.3) = 0.6 \quad (6) \end{aligned}$$

	S_1	S_2	S_3
A_1	1	0.3	0
A_2	0.8	0.6	0.3
A_3	0.6	0.6	0.6

Table 6: Utility table for decision maker A.

	S_1	S_2	S_3
A_1	1	0.15	0
A_2	0.6	0.3	0.15
A_3	0.3	0.3	0.3

Table 7: Utility table for decision maker B.

Thus, $EU = \max_i EU(A_i) = 0.6$. Decision maker A chooses investment A_3.

We now construct the utility table for decision maker B. This is Table 7. We calculate his expected utility for each action.

$$EU(A_1) = 1(0.4) + 0.15(0.3) + 0(0.3) = 0.445 \quad (7)$$

$$EU(A_2) = 0.6(0.4) + 0.3(0.3) + 0.15(0.3) = 0.375 \quad (8)$$

$$EU(A_3) = 0.3(0.4) + 0.3(0.3) + 0.3(0.3) = 0.3 \quad (9)$$

Thus, $EU = \max_i EU(A_i) = 0.445$. Decision maker B chooses investment A_1.

(c) **Why don't decision makers A and B select the same decision alternative?**

This highlights the subjective nature of the utility function and how it captures the individual's attitude to risk. Decision maker A is fairly risk averse, choosing investment A_3 with its guaranteed return. Decision maker B is less risk averse, choosing investment A_1 as he derives much greater pleasure from the maximum payoff than any other payoff (compare the utility of 0.80 for a payoff of \$75,000 for decision maker A to that of only 0.60 for decision maker B).