MA30118 - Solution Sheet One

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1. **Lincoln City are considering the number of programmes to print for Saturday's big match. They believe that there is a** 35% **chance that there will be a heavy turnout** (S_1)**, a** 50% **chance for a normal turnout** (S_2)**, and a** 15% **chance for a low turnout** (S_3)**. They must decide whether to print 500 copies** (A_1)**, 750 copies** (A_2)**, 1000 copies** (A_3)**, or 1250 copies** (A_4)**. The payoff table, Table 1, is given below. Unsold programmes would result in a loss.**

	S_1	S_2	S_3
A_1	100	100	100
A_2	150	140	110
A_3	200	160	75
A_4	250	120	-50

Table 1: Payoffs for programme production

(a) **Are all the actions admissible?**

The payoff for A_2 is always greater than that for A_1 whatever the state of nature. Thus, A_2 dominates A_1 and A_1 is inadmissible. We may remove A_1 from any further consideration.

(b) **Find the minimax regret decision.**

We find the best payoffs under each state of nature. Thus, $\pi^*(S_1) = 250$, $\pi^*(S_2) = 160$ and $\pi^*(S_3) = 110$. The corresponding opportunity regrets are shown in Table 2. From Table 2 we find the largest opportunity regret for each action. Thus,

	S_1	S_2	S_3
A_2	250 - 150 = 100	160 - 140 = 20	110 - 110 = 0
A_3	250 - 200 = 50	160 - 160 = 0	110 - 75 = 35
A_4	250 - 250 = 0	160 - 120 = 40	110 - (-50) = 160

Table 2: Opportunity regrets for programme production

Action	Maximum opportunity regret
A_2	100
A_3	50
A_4	160

The minimax regret decision is the one that minimises these. In this case, the minimax regret decision is A_3, we print 1000 copies.

(c) **Find the decision which maximises expected payoff.**

We calculate the expected monetary value for each three of the admissible actions, where, as there are three states of nature,

$$EMV(A_i) = \sum_{j=1}^{3} \pi(A_i, S_j)P(S_j).$$

Thus,

$$\begin{aligned} EMV(A_2) &= 150(0.35) + 140(0.50) + 110(0.15) &= 139, \\ EMV(A_3) &= 200(0.35) + 160(0.50) + 75(0.15) &= 161.25, \\ EMV(A_4) &= 250(0.35) + 120(0.50) + (-50)(0.15) &= 140. \end{aligned}$$

Hence, $EMV = max\{139, 161.25, 140\} = 161.25$. The decision which maximises EMV is A_3, we print 1000 copies.

2. **A manager is considering his staffing needs and has the following strategies: Lay off two staff employees (A_1); maintain staff at current levels (A_2); increase the staff size by one employee (A_3); increase the staff size by two employees (A_4). There are four possible states of nature: Business will decrease (S_1); business will stay the same (S_2); business will increase moderately (S_3); and business will increase rapidly (S_4). The prior probabilities are $P(S_1) = 0.15$, $P(S_2) = 0.25$, $P(S_3) = 0.25$ and $P(S_4) = 0.35$. The possible payoffs, in pounds as profits, are given in Table 3 overleaf. Find the decision**

	S_1	S_2	S_3	S_4
A_1	1451	1840	2050	2300
A_2	-1091	1685	2430	2900
A_3	-2015	1100	3060	3561
A_4	-3460	-1350	3340	4300

Table 3: Payoffs for the choice of staffing levels.

which maximises expected payoff.

We calculate the EMV for each action.

$$\begin{aligned} EMV(A_1) &= 1451(0.15) + 1840(0.25) + 2050(0.25) + 2300(0.35) &= 1995.15 \\ EMV(A_2) &= -1091(0.15) + 1685(0.25) + 2430(0.25) + 2900(0.35) &= 1880.1 \\ EMV(A_3) &= -2015(0.15) + 1100(0.25) + 3060(0.25) + 3561(0.35) &= 1984.1 \\ EMV(A_4) &= -3460(0.15) - 1350(0.25) + 3340(0.25) + 4300(0.35) &= 1483.5. \end{aligned}$$

Thus, $EMV = \max_i EMV(A_i) = 1995.1$ under action A_1: we lay off two staff employees.

3. **A firm decides to build a new factory which will last 10 years. Let h_1 denote the event that the demand is high for the first two years, l_1 the event that the demand is low for the first two years, h_2 that the demand is high for the last eight years and l_2 that the demand is low for the last eight years. These are the only possible outcomes. Marketing information reveals that $P(h_1, h_2) = P(h_1 \cap h_2) = 0.6$, $P(h_1, l_2) = P(h_1 \cap l_2) = 0.1$, $P(l_1, l_2) = P(l_1 \cap l_2) = 0.3$. Find $P(h_1)$, $P(h_2|h_1)$ and $P(l_2|h_1)$.**

   As demand for the final eight years can either be high (h_2) or low (l_2), then

$$P(h_1) \;=\; P(h_1 \cap h_2) + P(h_1 \cap l_2) \;=\; 0.6 + 0.1 \;=\; 0.7.$$

   Note that since demand for the first two years is either high (h_1) or low (l_1), it follows immediately that

$$P(l_1) \;=\; 1 - P(h_1) \;=\; 0.3,$$

   which, since we can't follow initial low demand with high demand, agrees with $P(l_1 \cap l_2)$. Using the definition of conditional probability,

$$\begin{aligned} P(h_2|h_1) &= \frac{P(h_1 \cap h_2)}{P(h_1)} = \frac{0.6}{0.7} = \frac{6}{7}, \\ P(l_2|h_1) &= \frac{P(h_1 \cap l_2)}{P(h_1)} = \frac{0.1}{0.7} = \frac{1}{7}. \end{aligned}$$

   Notice that, given h_1, demand for the last eight years is either h_2 or l_2 so that $P(h_2|h_1) + P(l_2|h_1) = 1$.

4. **The firm in question 3. may either build a big factory or a small factory which may then be expanded. If it builds a big factory and demand is high for the full ten years then the profit is £14million. If it builds big and demand is high for the first two years and low for the remaining eight then the profit is $-$£0.4million. A big factory with low demand for the entire ten years gives a profit of $-$£4million. If the firm builds a small factory and demand is high for the first two years then it may expand the factory. Expansion followed by eight years of high demand yields a profit of £6million, while expansion followed by eight years of low demand results in a profit of -£4.4million. If the factory is not expanded and demand remains high, then the profit is £3.2million, while if the demand falls to low for the remaining eight years then the profit is £5.6million. A small plant with low demand for the whole ten years returns a profit of £5.4million.**

   (a) **Construct the decision tree for the firm.**

   This is given in Figure 1.

   (b) **Find the expected value of perfect information (EVPI) for the firm.**

   We first find the expected monetary value under certainty, $EMVUC$. For the whole ten years, there are three possible outcomes, $h_1 \cap h_2$, $h_1 \cap l_2$ and $l_1 \cap l_2$.

Figure 1: The decision tree for the factory building question

Now,

$$\begin{aligned}
\pi^*(h_1 \cap h_2) &= \max\{14, 6, 3.2\} \ = \ 14, && (1)\\
\pi^*(h_1 \cap l_2) &= \max\{-0.4, -4.4, 5.6\} \ = \ 5.6, && (2)\\
\pi^*(l_1 \cap l_2) &= \max\{-4, 5.4\} \ = \ 5.4. && (3)
\end{aligned}$$

Thus,

$$\begin{aligned}
EMVUC &= \pi^*(h_1 \cap h_2)P(h_1 \cap h_2) + \pi^*(h_1 \cap l_2)P(h_1 \cap l_2) + \pi^*(l_1 \cap l_2)P(l_1 \cap l_2)\\
&= 14(0.6) + 5.6(0.1) + 5.4(0.3) \ = \ 10.58.
\end{aligned}$$

From Figure 1, $EMV = 7.16$, so that

$$EVPI \ = \ EMVUC - EMV \ = \ 10.58 - 7.16 \ = \ 3.42.$$

(c) **Which decision maximises the expected payoff. Comment on this choice of action.**

We should choose to build a large plant. Notice that having taken this decision, we will get a payoff of £14million with probability 0.6, −£0.4million with probability 0.1 and −£4million with probability 0.3. There is a (large) probability of 0.4 that we will lose money. The EMV of this decision is dominated by the large payoff of £14million. The EMV takes no account of the firms' ability to absorb any loss, and thus its' attitude to risk.