MA30118: MANAGEMENT STATISTICS
Assessed Coursework: Quality Control

1. (a) For each j,

$$\overline{x}_j = \frac{1}{5}\sum_{i=1}^{5} x_{ji} \quad \text{and} \quad r_j = \max_i(x_{ji}) - \min_i(x_{ji}).$$

Hence,

$$\begin{aligned}
\overline{x}_{21} &= \tfrac{170.9744}{5} = 34.19488, & r_{21} &= 34.2240 - 34.1760 = 0.0480,\\
\overline{x}_{22} &= \tfrac{171.0592}{5} = 34.21184, & r_{22} &= 34.2288 - 34.2000 = 0.0288,\\
\overline{x}_{23} &= \tfrac{170.9872}{5} = 34.19744, & r_{23} &= 34.2192 - 34.1728 = 0.0464,\\
\overline{x}_{24} &= \tfrac{170.9984}{5} = 34.19968, & r_{24} &= 34.2144 - 34.1808 = 0.0336,\\
\overline{x}_{25} &= \tfrac{170.9856}{5} = 34.19712, & r_{25} &= 34.2272 - 34.1712 = 0.0560.
\end{aligned}$$

[5]

(b) From lecture notes, the control limits for the R-chart with 3-sigma control limits are

$$\begin{aligned}
\text{UCL} &= D_4(n)\overline{r}\\
\text{CL} &= \overline{r}\\
\text{LCL} &= D_3(n)\overline{r}
\end{aligned}$$

where $\overline{r} = \frac{1}{25}\sum_{j=1}^{25} r_j$ and $D_3(n)$ and $D_4(n)$ are found from the attached table. As $n = 5$, we have $D_3(5) = 0$ (as the range is always non-negative and 3-sigma away will be negative), $D_4(5) = 2.115$. From Table 1 we find that

$$\overline{r} = \frac{0.9104}{25} = 0.036416.$$

Thus,

$$\begin{aligned}
\text{UCL} &= 2.115(0.036416) = 0.07701984\\
\text{CL} &= 0.036416\\
\text{LCL} &= 0.
\end{aligned}$$

The R-chart is shown in Figure **??**. When the 25 sample ranges are plotted on this chart, all of these are well inside the control limits. We conclude that the process variability appears to be in control (there also appears to be no obvious structure in the chart). [8]

(c) As the R-chart indicates that the process variability is in control, we are happy with $\overline{r}/d_2(5)$ (where $d_2(5) = 2.326$, as given on the attached table) as our unbiased estimate for the process standard deviation σ. From lecture notes, the control limits for the $\overline{x}$-chart with 3-sigma control limits are

$$\begin{aligned}
\text{UCL} &= \overline{\overline{x}} + A_2(n)\overline{r}\\
\text{CL} &= \overline{\overline{x}}\\
\text{LCL} &= \overline{\overline{x}} - A_2(n)\overline{r}
\end{aligned}$$

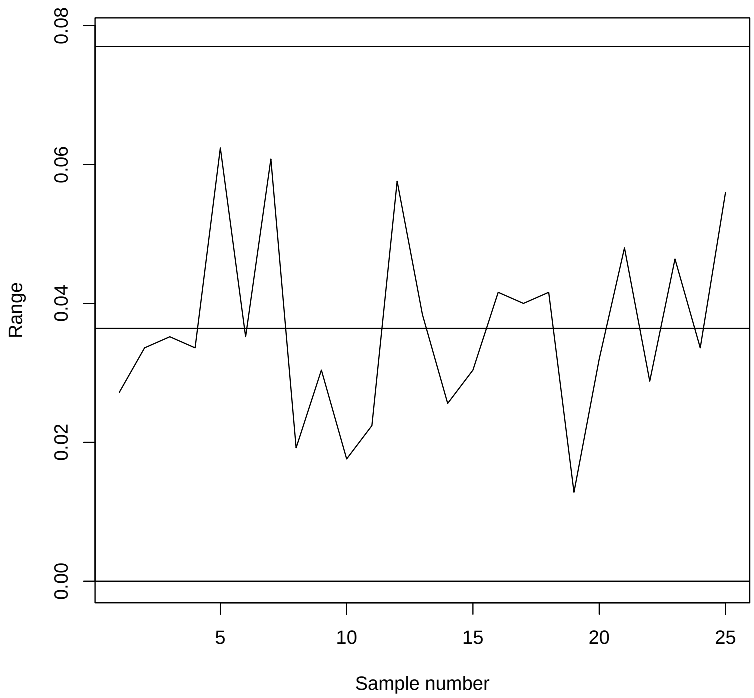

Figure 1: The R-chart for the data in Table 1.

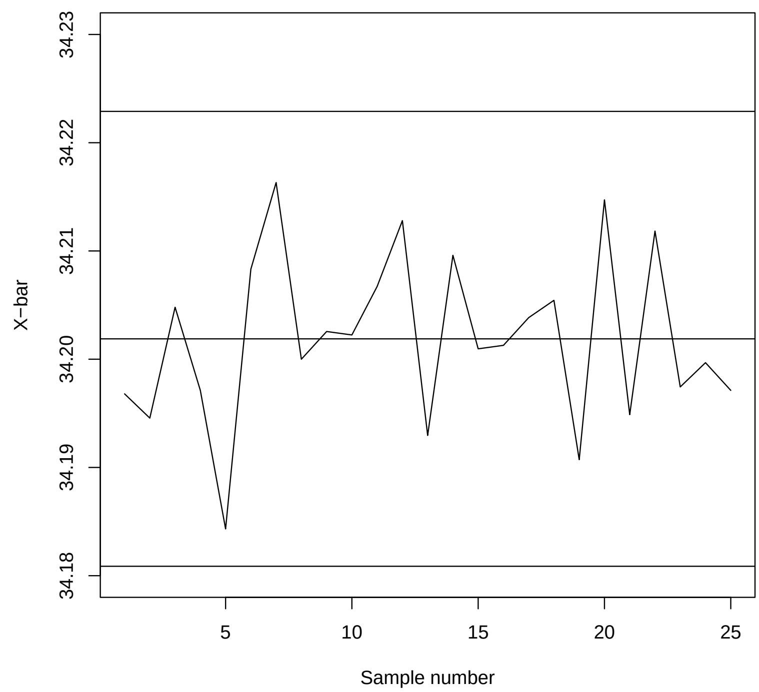

Figure 2: The $\overline{x}$-chart for the data in Table 1.

where $\overline{\overline{x}} = \frac{1}{25}\sum_{j=1}^{25} \overline{x}_j$ and $A_2(n)$ is found from the attached table. For $n = 5$ we have $A_2(5) = 0.577$ and

$$\overline{\overline{x}} = \frac{855.04704}{25} = 34.2018816.$$

Thus,

$$\begin{aligned} \text{UCL} &= 34.2018816 + 0.577(0.036416) = 34.22289363 \\ \text{CL} &= 34.2018816 \\ \text{LCL} &= 34.2018816 - 0.577(0.036416) = 34.18086957. \end{aligned}$$

The $\overline{x}$-chart is shown in Figure **??**. When the 25 sample means are plotted on this chart, all of these are well inside the control limits: there is no evidence of an out of control situation (additionally, there appears to be no structure to the chart). We are happy with $\overline{\overline{x}}$ as our unbiased estimate for the process mean. As both the R-chart and the $\overline{x}$-chart exhibit control, we conclude that the process is in control and we may use the control limits for monitoring the process in the future. [8]

(d) Using the $\overline{x}$-chart and R-chart developed in parts (b) and (c), we monitor the process by plotting the additional samples on the charts. These are shown in Figure **??**. The control charts suggest that the process is in control until $\overline{x}_{37}$ is

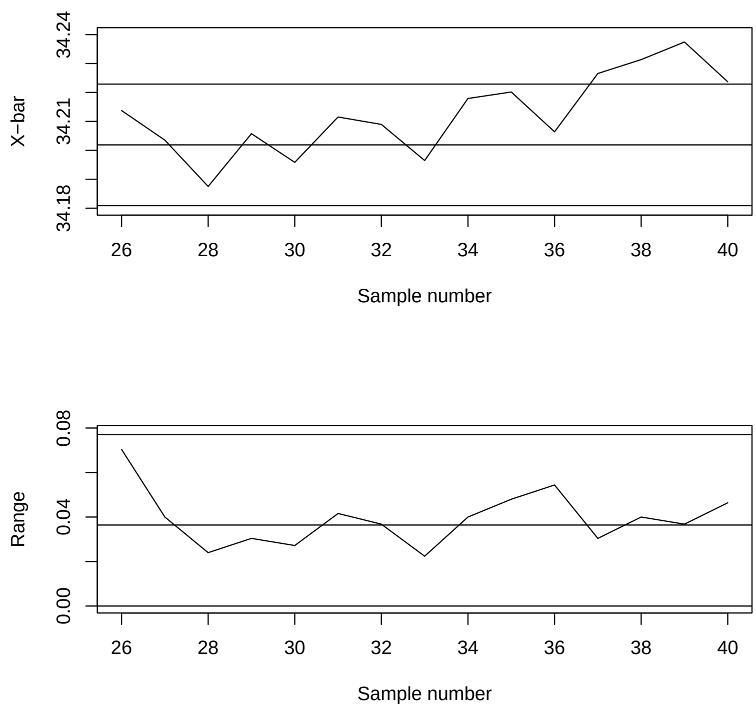

Figure 3: The $\overline{x}$-chart and R-chart for the additional samples given in Table 2.

plotted which crosses the upper control limit. We would suspect that there was an assignable cause of variation present and action should be taken to identify and rectify the cause. [10]

(e) The pattern of points on the $\overline{x}$-chart in Figure **??** could be explained by a shift in the mean of the process from about the 34th sample. Improving the sensitivity of the control charts by adding warning limits may alert us to any such shift earlier. (Note, the R-chart might suggest some centre-line hugging and that the variation of the process is smaller than that we have estimated). For the $\overline{x}$-chart,

$$\text{UCL - CL} = 34.22289363 - 34.2018816 = 0.02101203$$

so that

$$\begin{aligned} \text{UWL} &= 34.2018816 + \frac{2}{3}(0.02101203) = 34.21588962, \\ \text{LWL} &= 34.2018816 - \frac{2}{3}(0.02101203) = 34.18787358. \end{aligned}$$

For the R-chart,

$$\text{UCL - CL} = 0.07701984 - 0.036416 = 0.04060384$$

so that

$$\begin{aligned} \text{UWL} &= 0.036416 + \frac{2}{3}(0.04060384) = 0.063485226, \\ \text{LWL} &= \max\{0, 0.036416 - \frac{2}{3}(0.04060384)\} = 0.009346773. \end{aligned}$$

We now add these warning limits to the charts in Figure **??**. The corresponding charts are shown in Figure **??**. Note that in the $\overline{x}$-chart, we have two successive points: $\overline{x}_{34}$ and $\overline{x}_{35}$ outside the upper warning limit. We should stop after the 35th sample and suspect that there was an assignable cause of variation present. We detect that the process was potentially out of control earlier by adding the warning limits. [5]

(f) From the data in Table 1, our unbiased estimate of the process mean μ is $\overline{\overline{x}} = 34.2018816$ and of the process standard deviation σ is $\overline{r}/d_2(5) = 0.036416/2.326$. The $\overline{x}$-chart when μ and σ are known and samples are of size four are to be plotted is

$$\begin{aligned} \text{UCL} &= \mu + 3\frac{\sigma}{\sqrt{4}} \\ \text{CL} &= \mu \\ \text{LCL} &= \mu - 3\frac{\sigma}{\sqrt{4}} \end{aligned}$$

Replacing μ by $\overline{\overline{x}}$ and σ by $\overline{r}/d_2(5)$ we have

$$\begin{aligned} \text{UCL} &= \overline{\overline{x}} + 3\frac{\overline{r}}{d_2(5)\sqrt{4}} \\ \text{CL} &= \overline{\overline{x}} \\ \text{LCL} &= \overline{\overline{x}} - 3\frac{\overline{r}}{d_2(5)\sqrt{4}} \end{aligned}$$

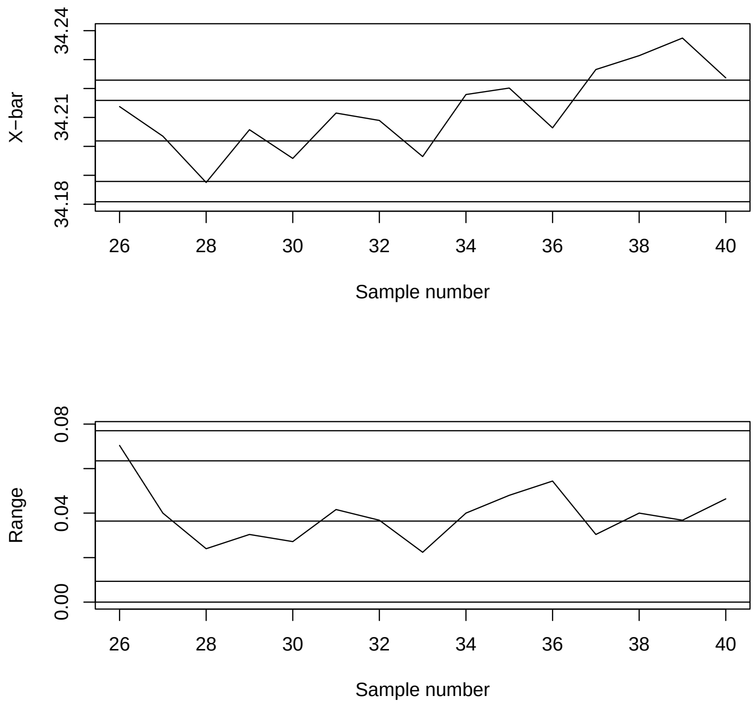

Figure 4: The $\overline{x}$-chart and R-chart for the additional samples given in Table 2 with warning limits added.

Thus, our limits are

$$\begin{aligned} \text{UCL} &= 34.2018816 + 3\frac{0.036416}{2.326 \times 2} = 34.22536569 \\ \text{CL} &= 34.2018816 \\ \text{LCL} &= 34.2018816 - 3\frac{0.036416}{2.326 \times 2} = 34.17839751 \end{aligned}$$

Notice that by reducing the sample size from five to four we widen the control limits as the sample means now have more variance due to the smaller sample size. This can be more clearly seen by noting that $A_2(n) = \frac{3}{d_2(n)\sqrt{n}}$ so that the modified $\overline{x}$-chart may be expressed as

$$\begin{aligned} \text{UCL} &= \overline{\overline{x}} + \sqrt{\frac{5}{4}} A_2(5)\overline{r} = \overline{\overline{x}} + \frac{d_2(4)}{d_2(5)} A_2(4)\overline{r} \\ \text{CL} &= \overline{\overline{x}} \\ \text{LCL} &= \overline{\overline{x}} - \sqrt{\frac{5}{4}} A_2(5)\overline{r} = \overline{\overline{x}} - \frac{d_2(4)}{d_2(5)} A_2(4)\overline{r} \end{aligned}$$

The R-chart when μ and σ are known and samples are of size four are to be plotted is

$$\begin{aligned} \text{UCL} &= \sigma d_2(4) + 3\sigma d_3(4) \\ \text{CL} &= \sigma d_2(4) \\ \text{LCL} &= \sigma d_2(4) - 3\sigma d_3(4) \end{aligned}$$

Replacing σ by $\overline{r}/d_2(5)$ we have

$$\begin{aligned} \text{UCL} &= \frac{d_2(4)}{d_2(5)}\overline{r} + 3\frac{d_3(4)}{d_2(5)}\overline{r} = \frac{d_2(4)}{d_2(5)} D_4(4)\overline{r} \\ \text{CL} &= \frac{d_2(4)}{d_2(5)}\overline{r} \\ \text{LCL} &= \frac{d_2(4)}{d_2(5)}\overline{r} - 3\frac{d_3(4)}{d_2(5)}\overline{r} = \frac{d_2(4)}{d_2(5)} D_3(4)\overline{r} \end{aligned}$$

Thus, our limits are

$$\begin{aligned} \text{UCL} &= \frac{2.059}{2.326} 2.282(0.036416) = 0.073562167 \\ \text{CL} &= \frac{2.059}{2.326}(0.036416) = 0.032235831 \\ \text{LCL} &= 0 \end{aligned}$$

Note that reducing the sample size lowers both the UCL and the CL which reflects the fact that the expected range from a sample of size four is smaller than that from a sample of size five. [10]

2. I marked this using a mixture of holistic and structured marking. There are nine broad areas I wanted you to touch upon with each being allocated six marks. The areas were

 - Introduction and interpretation

- Quality of argument and overall level of understanding
- Awareness of the assumptions underlying the construction of the control chart
- Simple methodologies for identifying when a process might be out of control and the limitations of this
- Discussion of an alternative to the perspective of a process being either "in control" or "out of control"
- Understanding of time series models and autocorrelation
- Use of the model; common cause chart and special cause chart
- Discussion of the methodology of the paper and examples therein
- Pros and cons of the methodology; simplicity of control charts versus complexity of time series modelling, validity of models

[54]