Quality Control

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1 Introduction

In quality control, we are concerned with problems involved in controlling the quality of a manufactured product. We'll study two major techniques:

1. ACCEPTANCE SAMPLING where we are concerned with monitoring the quality of manufactured items supplied by the manufacturer to consumers in batches. The problem is to decide whether the batch should be accepted or rejected on the basis of a sample randomly drawn from the batch.

2. PROCESS CONTROL where goods are produced continuously and the problem is to detect changes in the performance of the manufacturing process and take action (when necessary) to control the process.

1.1 What is quality?

Most of us associate quality with luxury (such as a BMW car, a plasma screen television, . . .) but by quality control we are thinking in terms of "things that work in the way we expect them to" i.e. "it does what it says on the tin".

1. Quality implies fitness for use.

2. Quality means conformance to requirements.

Thus, quality is defined by both customers and producers.

1.2 Variability compromises quality

Mass produced items are not identical. Some variation is inevitable and can cause problems. Too much variation might mean that parts which should fit together don't.

> e.g. A screw might be too small/large to fit the corresponding bolt.

There is a need to identify items which exhibit too much variation and deal with them, perhaps even scrap them.

1.3 Inspection versus Prevention

How should we deal with variation? An early strategy was to inspect goods at the end of the production line and release, to the consumer, only those of a sufficiently high standard. It was argued that the marginal cost of each item was small, so the cost of the rejected items was small. However, this argument ignored a number of other costs. For example, there is the cost of employing and equipping the inspectors; the damage to reputation if defective goods slip through the net and are released, and so on.

An alternative is that it is cheaper to do things right the first time, that is, to aim for zero defects.

> e.g. we demand perfection from electricity suppliers and airlines, why not from all producers?

When poorly made parts are passed down the production line, all subsequent work is wasted when the final product is rejected by the inspectors. It is expensive to keep inspecting components. It may be cheaper to try to prevent defects at each stage of manufacturing a product or delivering a service. At each stage, we could let production staff be responsible for checking the quality of their own work. This has the added benefit of giving workers a greater investment of pride in their work, making them more like craftsmen.

2 Acceptance sampling

Any manufacturing process will inevitably produce some defective items. The manufactured items will often by supplied in BATCHES or LOTS and may be examined by the manufacturer before shipment or by the consumer before acceptance.

Acceptance sampling is used when the cost of inspecting each item is too costly or may destroy the item, for example with chocolate bars. It is such problems we focus attention upon and not those where inspection of every item occurs.

> e.g. Precision engineering. F1 teams constantly check every part of the car to try to maximise performance and prevent failure on race-day.

When a batch is rejected by a sampling scheme, it may be returned, purchased at a lower cost or destroyed. Alternatively, it may be subjected to 100% inspection where all defective items are replaced by good items. This is called a rectifying scheme and we shall consider such schemes later.

Acceptance sampling techniques are applications of sampling theory and hypothesis testing and can be divided into two classes

1. SAMPLING BY ATTRIBUTE (qualitative approach). Here, items in the sample are classed dichotomously as 'good' or 'defective'.

2. SAMPLING BY VARIABLE (quantitative approach). Here, an items quality is determined as a result of a quantitative scale.

   e.g. A light bulb either works or it doesn't (qualitative). However, you might be interested in how many hours the bulb will last for (quantitative).

The general principles of the methodology are shared by both methods. Sampling by variable is usually more informative, but is also more costly.

e.g. You can just switch on a bulb to see whether it works or not, rather than continuously monitor its lifetime.

We shall concentrate upon sampling by attribute and look at three sampling schemes: single sampling, double sampling, and sequential sampling.

2.1 Single sampling scheme

In a single sampling scheme, we specify two numbers: n, the sample size and c, the acceptance number , the maximum allowable number of items with defects. The single sampling scheme is

$$\left.\begin{array}{l} \text{Select } n \text{ items,} \\ \text{Accept batch if number of defective items} \leq c, \\ \text{Reject batch if number of defective items} \geq c+1 \end{array}\right\}$$

Suppose that the entire batch consists of N items of which k are defective. The proportion of defective items $p = k/N$ is a measure of the quality of the batch. The higher the value of p, the worse the quality. Suppose we draw a sample of size n and let X be the number of defective items in the sample. Then,

$$P(X = d|N, k) \quad = \quad \frac{\left(\begin{array}{c} k \\ d \end{array}\right)\left(\begin{array}{c} N-k \\ n-d \end{array}\right)}{\left(\begin{array}{c} N \\ n \end{array}\right)}.$$

This is the hypergeometric distribution. If N is very much larger than n and the proportion $p = k/N$ is stable then X is approximately distributed as the binomial distribution with parameters n and p, written $Binomial(n, p)$. We shall assume this approximation is valid, so that

$$P(X = d|p) \quad = \quad \left(\begin{array}{c} n \\ d \end{array}\right) p^d (1-p)^{n-d}.$$

Hence,

$$P(\text{Reject batch}|p) \quad = \quad P(X \geq c+1|p) \quad = \quad \sum_{d=c+1}^{n} \left(\begin{array}{c} n \\ d \end{array}\right) p^d (1-p)^{n-d}$$

and

$$P(\text{Accept batch}|p) \quad = \quad P(X \leq c|p) \quad = \quad \sum_{d=0}^{c} \left(\begin{array}{c} n \\ d \end{array}\right) p^d (1-p)^{n-d}$$

$$= \quad 1 - P(\text{Reject batch}|p).$$

2.2 The operating characteristic curve

For ANY sampling scheme, $OC(p) = P(\text{Accept batch}|p)$ is known as the **operating characteristic (OC)** of the scheme. The plot of $OC(p)$ is the OC-curve.

e.g. for the single sampling scheme, $OC(p) = \sum_{d=0}^{c} \left(\begin{array}{c} n \\ d \end{array}\right) p^d (1-p)^{n-d}$ is the OC.
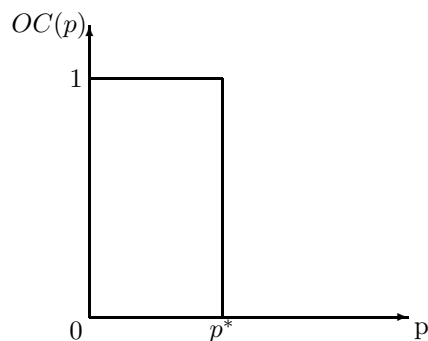
Figure 1: An ideal OC-curve.

If p is very small, we would like the batch to be accepted, while if p is large, we would like to reject the batch. When $p = 0$ there are no defective items in the batch and so the batch is certain to be accepted, thus $OC(0) = 1$. When $p = 1$ then all items are defective and so the batch is certain to be rejected, thus $OC(1) = 0$.

e.g. Suppose in a single sampling scheme that $n = 10$ and $c = 0$. Then,

$$OC(p) \;=\; \binom{10}{0} p^0 (1-p)^{10} \;=\; (1-p)^{10}.$$

If $n = 10$ and $c = 1$ then

$$OC(p) \;=\; (1-p)^{10} + \binom{10}{1} p^1 (1-p)^9 \;=\; (1+9p)(1-p)^9.$$

Thus, for a single sampling scheme, our choices of n and c affect the shape of the OC-curve. How should we choose n and c? In general, what shape should our OC-curve be like?

Ideally, there might be a critical value p^* such that we accept all batches with $p < p^*$ and reject all batches with $p^* < p$. That is,

$$OC(p) \;=\; \begin{cases} 1 & p < p^* \\ 0 & p > p^* \end{cases}$$

and is shown in Figure 1. This OC-curve is impossible to achieve without 100% inspection. An alternative would be to set

$$OC(p) \;=\; \begin{cases} 1 & p < p_1 \\ 0 & p > p_2 \end{cases} \quad p_2 > p_1$$

leaving the region (p_1, p_2) in which we do not mind what happens. Such an OC-curve is shown in Figure 2. Unfortunately, this is also impossible without 100% inspection. So, we wish to devise a sampling plan so that if p is acceptably small then there is a high probability of accepting this batch of 'good' quality and if p is unacceptably large then there is a small probability of accepting this batch of 'bad' quality. The producer and consumer should get together and decide upon a sampling plan which is fair to both.

Definition 1 *(AQL and producer's risk)*
The largest proportion of defectives in a batch which is acceptable to the consumer is called the acceptable quality level (AQL) and is denoted by p_1. The producer's risk is $P(Reject\ batch|p = p_1) = \alpha$ so that $OC(p_1) = 1 - \alpha$.
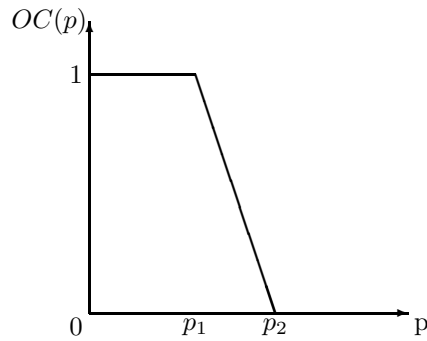
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Figure 2: An alternative form of an ideal OC-curve.

Thus, the producer's risk is the probability of rejecting a batch at the limit of acceptable quality.

> i.e. the producer doesn't shift a batch that the consumer judges acceptable.

In the idealised scheme shown in Figure 2 we have $\alpha = 0$.

Definition 2 *(LTPD and consumer's risk)*
The smallest proportion of defectives in a batch which is unacceptable to the consumer is called the unacceptable quality level and is denoted by p_2. The corresponding percentage, $100p_2$, is termed the lot tolerance percentage defective (LTPD). The consumer's risk is $P(Accept\ batch|p = p_2) = \beta$ so that $OC(p_2) = \beta$.

Thus, the consumer's risk is the probability of accepting a batch at the lower limit of bad quality.

> i.e. the consumer accepts a batch that he judges unacceptable.

In the idealised scheme shown in Figure 2 we have $\beta = 0$.

The producer's and consumer's risks are illustrated in Figure 3. Notice that the producer's and consumer's risks specify two points on the OC-curve.

2.2.1   A link to hypothesis testing

If we consider testing the hypothesis $H_0 : p = p_1$ against the alternative $H_A : p > p_1$ then $1 - OC(p)$ is the corresponding power curve. Rejecting a batch in the sampling scheme is equivalent to rejecting H_0. If the batch is rejected when H_0 is actually true, then this is a type I error, and the probability of such an error is the producer's risk, α. Conversely, if we accept the batch when it should have been rejected, we have a type II error. When $p = p_2$, the probability of a type II error is the consumer's risk, β.

2.3   Calculating the OC-curve for single sampling schemes

2.3.1 n and c given

Bertie Bassett is negotiating a contract for buying batches of 1000 packets of Liquorice Allsorts from Sweets'R'Us. Each packet is judged good if it contains more than 20 Liquorice Allsorts and defective if there are less than 20 Liquorice Allsorts. Sweets'R'Us claim that it can produce batches with proportion of defective packets below 1% which is acceptable
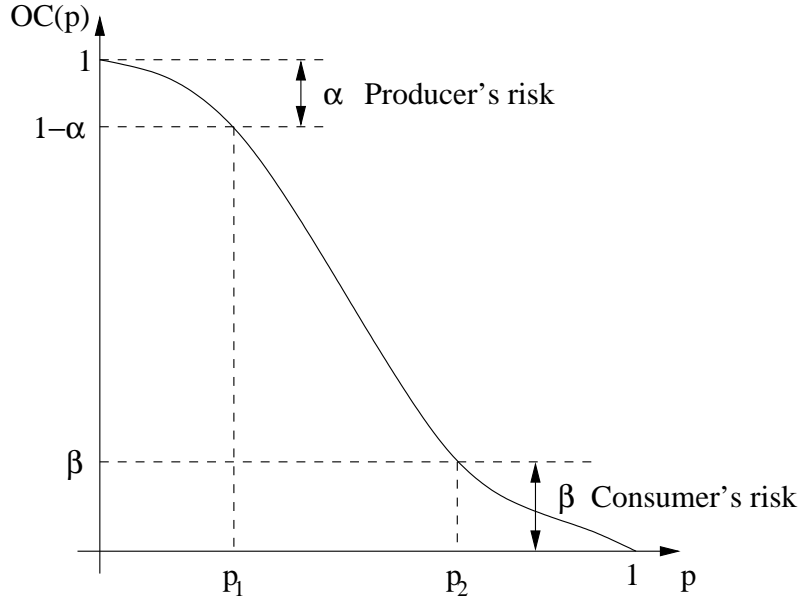
Figure 3: Producer's and consumer's risks.

to Bertie. The AQL is thus 0.01. Suppose Bertie chooses $n = 100$ and $c = 1$ in a single sampling scheme. He calculates the producer's risk.

$$
\begin{aligned}
\alpha &= P(\text{Reject batch}|p = 0.01) \\
&= 1 - \binom{100}{0}(0.01)^0(0.99)^{100} - \binom{100}{1}(0.01)^1(0.99)^{99} \\
&= 1 - 0.3660 - 0.3697 \\
&= 0.2643.
\end{aligned}
$$

The minimum defect rate Bertie would like to reject is 4%. This is the LTPD. He calculates the corresponding consumer's risk.

$$
\begin{aligned}
\beta &= P(\text{Accept batch}|p = 0.04) \\
&= \binom{100}{0}(0.04)^0(0.96)^{100} + \binom{100}{1}(0.04)^1(0.96)^{99} \\
&= 0.01687 + 0.07029 \\
&= 0.08716.
\end{aligned}
$$

The producer may not be willing to accept such a high risk, especially in comparison to the consumer's risk. What should they do? How should they choose mutually satisfying n and c? Notice that increasing c, keeping n fixed, will decrease the producer's risk and increase the consumer's risk. Conversely, increasing n, keeping c fixed, decreases the consumer's risk and increases the producer's risk.

2.3.2 Finding n and c from given producer's and consumer's risks

One way to choose n and c is to use specified producer's and consumer's risks at given AQL, p_1, and LTPD, $100p_2$. Notice that for p known, we may view the OC-curve as a function of

n and c. We would like to choose n and c such that

$$f_1(n, c) = P(\text{Reject batch}|p = p_1) \leq \alpha$$
$$f_2(n, c) = P(\text{Accept batch}|p = p_2) \leq \beta$$

Thus, we have two inequalities and two unknowns and so we can solve for n and c. Tables exist to help find solutions to these inequalities: these are attached to your first homework sheet. These are copies of Tables 3 and 6 from 'Sampling Inspection and Quality Control' by G. B. Wetherill, 2nd edition, Chapman & Hall, 1977. Table 3 gives the percentage points of the χ^2-distribution. We let $\chi^2_{p,v}$ denote the value of the χ^2-distribution with v degrees of freedom for which the probability of observing a value less than or equal to $\chi^2_{p,v}$ is p. Table 6 gives values of the ratio $r(c) = \chi^2_{1-\beta,2(c+1)}/\chi^2_{\alpha,2(c+1)}$.

(The actual solution of these inequalities is beyond the scope of this course. If you want a reference go to

www.york.ac.uk/depts/maths/teaching/pml/ais/acceptance.ps

then Section 1(f) page 8 onwards)

To find n and c is a three staged approach as we now illustrate.

Suppose Bertie is happy with a consumer's risk of 0.05 and Sweets'R'Us with a producer's risk of 0.05 with (as before) an AQL of 0.01 and LTPD of 4. Thus, $p_1 = 0.01$, $p_2 = 0.04$, $\alpha = 0.05$ and $\beta = 0.05$.

1. Calculate p_2/p_1. In this case, $p_2/p_1 = 0.04/0.01 = 4$.

2. Use Table 6 with the given values of α and β to find the smallest c such that

$$r(c-1) > \frac{p_2}{p_1} > r(c).$$

For our example, $\alpha = 0.5$ and $1 - \beta = 0.95$. So, we use the second column of Table 6(b). We see that $r(5) = 4.02$ and $r(6) = 3.60$ so that $4.02 > 4 > 3.60$. We thus choose $c = 6$.

3. The possible values of n are

$$\frac{\chi^2_{1-\beta,2(c+1)}}{2p_2} \leq n \leq \frac{\chi^2_{\alpha,2(c+1)}}{2p_1},$$

where Table 3 is used to find $\chi^2_{1-\beta,2(c+1)}$ and $\chi^2_{\alpha,2(c+1)}$. In our case, $2(c+1) = 2 \times 7 = 14$. From Table 3 we find $\chi^2_{0.95,14} = 23.68$ and $\chi^2_{0.05,14} = 6.57$. Thus, our possible values of n are

$$\frac{23.68}{2(0.04)} \leq n \leq \frac{6.57}{2(0.01)} \Rightarrow$$
$$296 \leq n \leq 328.5.$$

A particular solution is thus $c = 6$ and $n = 300$. Notice that we can check our work. With $p_1 = 0.01$, $p_2 = 0.04$, $c = 6$ and $n = 300$ then

$$\alpha = P(\text{Reject batch}|p = 0.01) = 0.03275$$
$$\beta = P(\text{Accept batch}|p = 0.04) = 0.04277$$
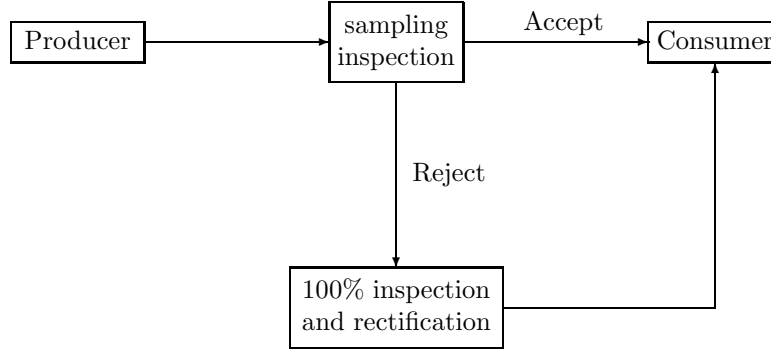
which conform with our requirements.

Figure 4: Illustration of a rectifying scheme.

2.4 Rectifying schemes

A common procedure when using single sampling is to subject any batch that has been rejected to 100% inspection of the items and to replace any defective items with good ones. Such a strategy is called a rectifying scheme and is illustrated in Figure 4. Thus, the consumer receives two types of batches:

1. Those which passed the inspection and may still contain defective items.

2. Perfect batches that have been rectified and defective items replaced having failed the sampling inspection.

The quality (proportion of defectives) in batches that the consumer receives is better than the quality of the input to the sampling inspection: the consumer has a positive probability of receiving some batches that contain no defects, whereas without rectifying will only receive batches with a proportion p of defects.

Definition 3 *(AOQ)*
The average outgoing quality (AOQ) is the average proportion of defectives in batches received by the consumer.

Thus,

$$
\begin{aligned}
AOQ(p) \quad &= \quad E(\text{Proportion defectives received}|\text{Accept batch}, p)P(\text{Accept batch}|p) + \\
&\quad\quad E(\text{Proportion defectives received}|\text{Reject batch}, p)P(\text{Reject batch}|p) \\
&= \quad p \times P(\text{Accept batch}|p) + 0 \times P(\text{Reject batch}|p) \\
&= \quad pOC(p) \leq p.
\end{aligned}
$$

Note that when $p = 0$, $AOQ(0) = 0 \times OC(0) = 0 \times 1 = 0$ and when $p = 1$, $AOQ(1) = 1 \times OC(1) = 1 \times 0 = 0$. As $AOQ(p)$ is a non-negative function then it thus has a maximum between 0 and 1.

Definition 4 *(AOQL)*
The average outgoing quality limit (AOQL) is the maximum of the AOQ.

Hence, whatever the input quality, p, the consumer knows that $AOQ \leq AOQL$ and so he knows the worse case scenario.
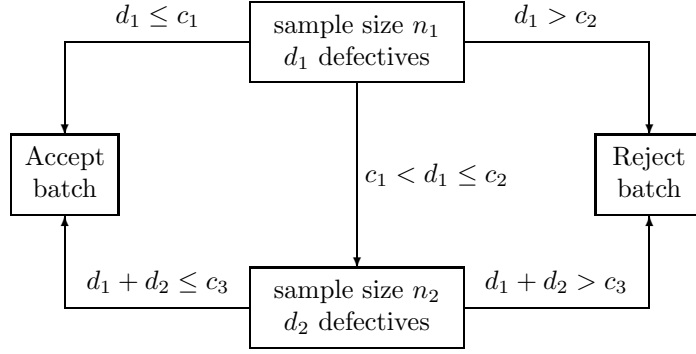
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Figure 5: Double sampling scheme.

2.5 Double sampling schemes

A simple extension of the single sampling scheme is obtained by a two-stage sampling procedure. A sample of size n_1 is drawn but the batch need not be accepted or rejected as a result of the first sample if it leaves some doubt as to the quality of the batch. A second sample of size n_2 may be drawn and the results combined before a decision is made. The scheme is illustrated in Figure 5 So, the basic idea is to cut down on the amount of sampling one needs to do by taking an initially small sample, n_1. If the number of defectives d_1 found in this sample are sufficiently small (less than or equal to c_1) then the batch is accepted. If d_1 is sufficiently large (greater than c_2) then the batch is rejected. If d_1 falls between these cut-off points then a second sample, of size n_2, is taken in which d_2 defective items are observed. The batch is then either accepted or rejected is the total number of defective items in both samples, $d_1 + d_2$ is less than or equal to, or exceeds, the critical value c_3.

The plan is often simplified by taking $c_2 = c_3$ so that there is a critical level of defective ideas we wish to see.

2.5.1 Operating characteristic for the double sampling scheme

We calculate the operating characteristic. Note that if we know the number of defectives in the first sample then we know whether the batch is immediately rejected or accepted or if the second sample is taken. If the second sample is taken, knowledge of the number of defectives in the first sample enables us to deduce the maximum number of defectives in the second second in order for us to accept the batch.

Let X_1 denote the number of defectives in the first sample and X_2 the number of defectives in the second sample. To calculate the operating characteristic we form a partition over the possible values of X_1 and use the theorem of total probability. Thus,

$$
\begin{aligned}
OC(p) \ &= \ P(\text{Accept batch}|p) \\
&= \ P(\text{Accept batch}|p, X_1 \leq c_1)P(X_1 \leq c_1|p) + \\
&\quad P(\text{Accept batch}|p, X_1 = c_1 + 1)P(X_1 = c_1 + 1|p) + \\
&\quad P(\text{Accept batch}|p, X_1 = c_1 + 2)P(X_1 = c_1 + 2|p) + \cdots + \\
&\quad P(\text{Accept batch}|p, X_1 = c_2)P(X_1 = c_2|p) + \\
&\quad P(\text{Accept batch}|p, X_1 > c_2)P(X_1 > c_2|p).
\end{aligned}
$$

Now, given that $X_1 \leq c_1$, the batch is accepted so that $P(\text{Accept batch}|p, X_1 \leq c_1) = 1$ while given that $X_1 > c_2$ the batch is rejected so that $P(\text{Accept batch}|p, X_1 > c_2) = 0$.

Thus,

$$OC(p) = P(X_1 \le c_1|p) + \sum_{d_1=c_1+1}^{c_2} P(\text{Accept batch}|p, X_1 = d_1)P(X_1 = d_1|p).$$

Now, for $d_1 \in \{c_1 + 1, \dots, c_2\}$ the second sample is taken and

$$
\begin{aligned}
P(\text{Accept batch}|p, X_1 = d_1) &= P(X_1 + X_2 \le c_3|p, X_1 = d_1) \\
&= P(X_2 \le c_3 - d_1|p, X_1 = d_1) \\
&= P(X_2 \le c_3 - d_1|p),
\end{aligned}
$$

where the last equation follows since, given p, the two samples are independent. Hence,

$$OC(p) = P(X_1 \le c_1|p) + \sum_{d_1=c_1+1}^{c_2} P(X_2 \le c_3 - d_1|p)P(X_1 = d_1|p).$$

We continue with the assumption that the batch size is sufficiently large that we may use the binomial sampling approach. Thus,

$$
\begin{aligned}
OC(p) = &\sum_{d_1=0}^{c_1} \binom{n_1}{d_1} p^{d_1}(1-p)^{n_1-d_1} + \\
&\sum_{d_1=c_1+1}^{c_2} \left\{ \sum_{d_2=0}^{c_3-d_1} \binom{n_2}{d_2} p^{d_2}(1-p)^{n_2-d_2} \right\} \binom{n_1}{d_1} p^{d_1}(1-p)^{n_1-d_1}.
\end{aligned}
$$

2.5.2 Worked example

Suppose that $n_1 = 60$, $n_2 = 100$, $c_1 = 0$ and $c_2 = c_3 = 2$. Thus, we will accept the batch without taking the second sample if $X_1 = 0$ and reject the batch without taking the second sample if $X_1 > 2$. If $X_1 = 1$ or $X_1 = 2$ then the second sample is taken. In the former case, the batch will be accepted if $X_2 \le 1$ and in the latter case acceptance will occur only if $X_2 = 0$. Thus,

$$OC(p) = P(X_1 = 0|p) + P(X_2 \le 1|p)P(X_1 = 1|p) + P(X_2 = 0|p)P(X_1 = 2|p).$$

So, if $p = 0.01$ then:

$$
\begin{aligned}
P(X_1 = 0|p = 0.01) &= \binom{60}{0}(0.01)^0(0.99)^{60} = 0.5472, \\
P(X_1 = 1|p = 0.01) &= \binom{60}{1}(0.01)^1(0.99)^{59} = 0.3316, \\
P(X_1 = 2|p = 0.01) &= \binom{60}{2}(0.01)^2(0.99)^{58} = 0.0988, \\
P(X_2 = 0|p = 0.01) &= \binom{100}{0}(0.01)^0(0.99)^{100} = 0.3660, \\
P(X_1 = 1|p = 0.01) &= \binom{100}{0}(0.01)^1(0.99)^{99} = 0.3697.
\end{aligned}
$$

So,

$$OC(0.01) = 0.5472 + (0.3660 + 0.3697)0.3316 + (0.3660)0.0988 = 0.827.$$

Similarly, for $p = 0.05$ we find

$$
\begin{aligned}
P(X_1 = 0|p = 0.05) &= 0.0461, \quad P(X_2 = 0|p = 0.05) = 0.0059, \\
P(X_1 = 1|p = 0.05) &= 0.1455, \quad P(X_2 = 1|p = 0.05) = 0.0312, \\
P(X_1 = 2|p = 0.05) &= 0.2259.
\end{aligned}
$$

Thus,

$$
OC(0.05) = 0.0461 + (0.0059 + 0.0312)0.1455 + (0.0059)0.2259 = 0.053.
$$

We would like to know how much sampling we'd expect to do.

Definition 5 *(ASN)*
The average sample number (ASN) for the double sampling scheme is the expected sample size.

Hence,

$$
\begin{aligned}
ASN(p) &= n_1 P(\text{Don't take second sample}|p) + (n_1 + n_2)P(\text{Take second sample}|p) \\
&= n_1\{P(X_1 \leq c_1|p) + P(X_1 > c_2|p)\} + (n_1 + n_2)P(c_1 < X_1 \leq c_2|p) \\
&= n_1 + n_2 P(c_1 < X_1 \leq c_2|p)
\end{aligned}
$$

Thus, $ASN(p) \leq n_1 + n_2$. For the double sampling scheme described above,

$$
ASN(p) = 60 + 100P(0 < X_1 \leq 2|p).
$$

For example, if $p = 0.01$,

$$
ASN(0.01) = 60 + 100(0.3316 + 0.0988) = 103.04.
$$

2.6 Sequential sampling

The amount of inspection can be reduced by using double sampling rather than taking the whole sample at once. We can go further than this and consider sampling items one at a time. We may be able to reach a decision before sampling the nth item where n is a fixed sample size. We shall explore a method called the **sequential probability ratio test (SPRT)**. Notice that at the first stage of the double sampling scheme, we partitioned the sample space into:

1. Acceptance region: accept batch if $d_1 \leq c_1$

2. Rejection region: reject batch if $d_1 \geq c_2 + 1$

3. Continue sampling region: take second sample if $c_1 < d_1 < c_2 + 1$

The idea behind sequential sampling is to generalise this to each individual observation. Consider the case where I inspect observations one by one and let n denote the cumulative sample size and d the number of defective items in these n samples. Clearly, $n \geq d$. The closer d is to n the more we suspect the proportion of defectives in the batch is high and so would like to reject the batch. Similarly, if d is close to 0 the more we suspect the proportion of defectives to be small and so would like to accept the batch.

Suppose we specify an AQL of p_1, a LTPD of $100p_2$ ($p_2 > p_1$), a producer's risk of α and a consumer's risk of β. The sequential sampling rule we adopt, the SPRT, is as follows; a derivation is given in §2.6.2.

1. Acceptance region: if $d \leq a + nc$ then accept the batch.

2. Rejection region: if $d \geq b + nc$ then reject the batch.

3. Continuation rule: if $a + nc < d < b + nc$ then continue sampling (until the batch is either accepted or rejected)

where

$$a = \frac{1}{k}log\left\{\frac{\beta}{1-\alpha}\right\}, \quad b = \frac{1}{k}log\left\{\frac{1-\beta}{\alpha}\right\}, \quad c = \frac{1}{k}log\left\{\frac{1-p_1}{1-p_2}\right\}, \quad k = log\left\{\frac{p_2(1-p_1)}{p_1(1-p_2)}\right\}$$

Note that as $p_2 > p_1$ then $1 - p_1 > 1 - p_2$. It is then straightforward to show that $0 < c < 1$.

We may calculate the earliest points at which we would accept or reject the batch. Notice that rejection of a batch always occurs after an observation of a defective item and acceptance will only occur after observation of an acceptable item. The first n at which rejection can occur if when we have observed only defective items, so $n = d$, and we are in the rejection region.

- The first n at which rejection can occur is given by

$$\min\{n : n \geq b + nc\} \text{ i.e. the first integer } n \text{ such that } n \geq \frac{b}{1-c}.$$

The first n at which acceptance can occur if when we have observed only acceptable items, so $d = 0$, and we are in the acceptance region.

- The first n at which acceptance can occur is given by

$$\min\{n : a + nc \geq 0\} \text{ i.e. the first integer } n \text{ such that } n \geq \frac{-a}{c}.$$

A diagrammatic representation of the SPRT is given in Figure 6.

2.6.1 Worked example

The Manager of Worldwide Laptop Computers has plans to introduce a quality control procedure for assessing batches of recently-manufactured computers. Computers are to be sampled from the batch sequentially and then subjected to a variety of tests. Depending on the outcome of these tests, the sampled computer will be classified as either *satisfactory* or *unsatisfactory*, and a decision is then made either to *stop* or *continue* sampling. You have been employed by the Manager to implement this sampling scheme, a sequential probability ratio test (SPRT) with $AQL = 0.1$, $LTPD = 20$, producer's risk $\alpha = 0.1$ and $\beta = 0.2$.

1. **Determine the lower and upper boundaries of this SPRT and explain how the SPRT is conducted.**

   Under the usual notation, we have $p_1 = 0.1$ and $p_2 = 0.2$. We calculate

   $$
   \begin{aligned}
   k &= \log\left\{\frac{p_2(1-p_1)}{p_1(1-p_2)}\right\} = \log\left(\frac{0.2 \times 0.9}{0.1 \times 0.8}\right) = 0.810930216; \\
   a &= \frac{1}{k}\log\left(\frac{\beta}{1-\alpha}\right) = \frac{1}{k}\log\left(\frac{0.2}{0.9}\right) = \frac{-1.504077397}{0.810930216} = -1.854755646;
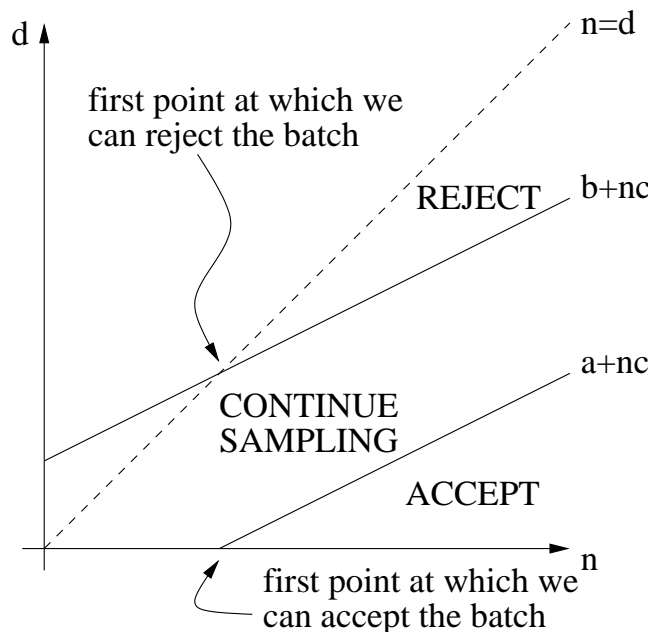   \end{aligned}
   $$

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Figure 6: A diagrammatic representation of the SPRT.

$$b = \frac{1}{k}\log\left(\frac{1-\beta}{\alpha}\right) = \frac{1}{k}\log\left(\frac{0.8}{0.1}\right) = \frac{2.079441542}{0.810930216} = 2.564266937;$$

$$c = \frac{1}{k}\log\left(\frac{1-p_1}{1-p_2}\right) = \frac{1}{k}\log\left(\frac{0.9}{0.8}\right) = \frac{0.117783035}{0.810930216} = 0.145244354.$$

We sequentially sample items at random from the batch. Let d be the number of defectives in the first n items. Our sequential sampling rule is:

- Acceptance region: if $d \leq -1.854755646 + 0.145244354n$ then accept the batch.
- Rejection region: if $d \geq 2.564266937 + 0.145244354n$ then reject the batch.
- Continuation rule: if

$$-1.854755646 + 0.145244354n < d < 2.564266937 + 0.145244354n$$

then continue sampling (until the batch is either accepted or rejected).

2. **What is the smallest number of observations at which the rejection of the batch can occur?**

The first n at which rejection can occur is $\min\{n : n \geq b + cn\}$, that is the smallest integer

$$n \geq \frac{b}{1-c} = \frac{2.564266937}{1 - 0.145244354} = 3.$$

So, the smallest number of observations at which rejection can occur is 3.
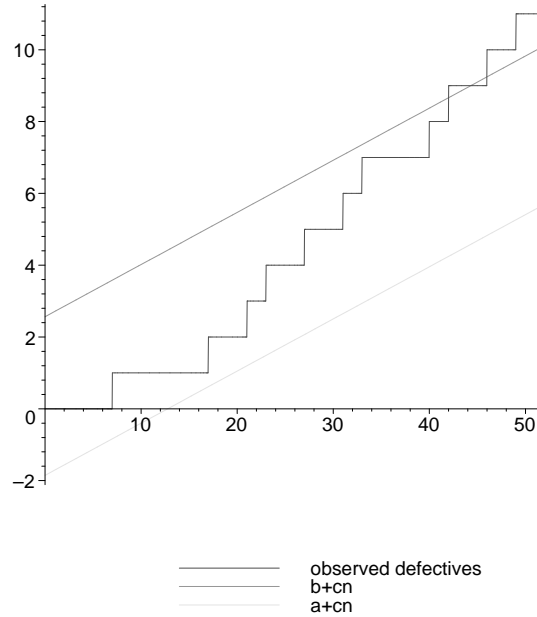
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Figure 7: Step-function and sampling boundaries for the SPRT.

3. **What is the smallest number of observations at which the acceptance of the batch can occur?**

   The first n at which acceptance can occur is given by $\min\{n : a + cn \geq 0\}$, that is the smallest integer

   $$n \geq \frac{-a}{c} = \frac{1.854755646}{0.145244354} = 12.76989842.$$

   So, the smallest number of observations at which acceptance can occur is 13.

4. **The sequential sampling procedure was applied to a batch of new computers. The computers which were examined at position numbers**

   $$7,\ 17,\ 21,\ 23,\ 27,\ 31,\ 33,\ 40,\ 42,\ 46,\ 49$$

   in the sequence were deemed unsatisfactory, and all the remaining computers in the sequence were considered satisfactory. [You should ignore the fact that, in practice, sampling would cease once a decision is made!] By plotting the number of defects against the sample size, carry out the SPRT on this data. State the outcome of the test.

   The step-function showing the number of observed defectives against the total number of samples is shown in Figure 7. We continue sampling until the step-function crosses either the upper and lower boundary upon which we cease sampling. We observe that we first cross a boundary, the upper boundary, when we observe the ninth defective item which is when $n = 42$. We stop and reject the batch after the 42nd sample.

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2.6.2 Derivation of the SPRT (Not examined)

Suppose that $p_2 > p_1$ are two values of interest for the proportion of defectives in the batch and consider the ratio

$$\lambda_n = \frac{P(d \text{ defectives in } n \text{ observations} \,|\, p = p_2)}{P(d \text{ defectives in } n \text{ observations} \,|\, p = p_1)}.$$

If λ_n is large then there is evidence to support the hypothesis $p = p_2$ while small values of λ_n support the hypothesis that $p = p_1$. Typically, we shall focus attention upon p_1 being the AQL and $100p_2$ being the LTPD. Thus, if λ_n is small we would like to accept the batch while if λ_n is large we would like to reject the batch. So, assume that there are critical values A and B such that:

$$\frac{P(d \text{ defectives in } n \text{ observations} \,|\, p = p_2)}{P(d \text{ defectives in } n \text{ observations} \,|\, p = p_1)} \geq B \quad \Rightarrow \quad \text{Reject batch}$$

$$\frac{P(d \text{ defectives in } n \text{ observations} \,|\, p = p_2)}{P(d \text{ defectives in } n \text{ observations} \,|\, p = p_1)} \leq A \quad \Rightarrow \quad \text{Accept batch}$$

If we are between A and B then we'd like to continue sampling. Compare this argument with the idealised OC-curves in Figures 1 and 2. Thus, we continue sampling as long as

$$A < \frac{P(d \text{ defectives in } n \text{ observations} \,|\, p = p_2)}{P(d \text{ defectives in } n \text{ observations} \,|\, p = p_1)} < B.$$

Using the respective binomial probabilities, we continue sampling as long as

$$A < \left\{ \frac{p_2}{p_1} \right\}^d \left\{ \frac{(1 - p_2)}{(1 - p_1)} \right\}^{n-d} < B \Leftrightarrow$$

$$A < \left\{ \frac{p_2(1 - p_1)}{p_1(1 - p_2)} \right\}^d \left\{ \frac{(1 - p_2)}{(1 - p_1)} \right\}^n < B \Leftrightarrow$$

$$\frac{log A + n log \left\{ \frac{1-p_1}{1-p_2} \right\}}{log \left\{ \frac{p_2(1-p_1)}{p_1(1-p_2)} \right\}} < d < \frac{log B + n log \left\{ \frac{1-p_1}{1-p_2} \right\}}{log \left\{ \frac{p_2(1-p_1)}{p_1(1-p_2)} \right\}}. \tag{1}$$

The constraint (1) is the **continuation rule**. How do we choose A and B?

If p_1 is the AQL and $100p_2$ the LTPD then to guarantee a producer's risk of less than α and a consumer's risk of less than β then (from Wald's theorem) we must choose

$$B \leq \frac{1 - \beta}{\alpha} \quad \text{and} \quad A \geq \frac{\beta}{1 - \alpha}.$$

In particular, we take $B = (1 - \beta)/\alpha$ and $A = \beta/(1 - \alpha)$. So, let

$$a = \frac{1}{k} log \left\{ \frac{\beta}{1 - \alpha} \right\}, \quad b = \frac{1}{k} log \left\{ \frac{1 - \beta}{\alpha} \right\}, \quad c = \frac{1}{k} log \left\{ \frac{1 - p_1}{1 - p_2} \right\}, \quad k = log \left\{ \frac{p_2(1 - p_1)}{p_1(1 - p_2)} \right\}$$

then our sequential sampling rule is

1. Acceptance region: if $d \leq a + nc$ then accept the batch.

2. Rejection region: if $d \geq b + nc$ then reject the batch.

3. Continuation rule: if $a + nc < d < b + nc$ then continue sampling (until the batch is either accepted or rejected).

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3 Process Control

3.1 Introduction

In any production process, some variation in quality is unavoidable. This variation can be divided into two categories:

1. Inherent random variation which is characteristic of the process and cannot be entirely eliminated. This is sometimes called variation due to **allowable causes**.

2. Variation due to specific identifiable causes such as malfunction, incorrect setting, wear and tear, . . . These are **assignable causes of variation**.

When only allowable causes of variation are present, the process is said to be IN CONTROL. If variation due to one or more assignable cause is present, the process is said to be OUT OF CONTROL.

The objectives of process control are

1. To detect any changes due to assignable causes as early as possible (and hence, try to identify the cause and take appropriate action to rectify the problem).

2. To allow production to continue without too many interruptions when the process is in control.

3.2 Control charts

The most commonly used tool for detecting changes is the CONTROL CHART (or SHEWHART CHART after its developer Walter Shewhart). Many quality characteristics can be expressed in terms of a numerical measurement or variable

e.g. dimension, weight, volume, . . .

When dealing with a quality characteristic that is a variable it is usually necessary to monitor both the mean value of the quality characteristic and its variability.

- Control of the process average, or mean quality level, is usually achieved with the $\overline{x}$-chart, the control chart for means.

- Process variability can be monitored by either a control chart for the standard deviation, the S-chart, or a control chart for the range, the R-chart. The R-chart is more widely used, largely because of the simplicity of calculating the range.

- Typically, separate $\overline{x}$ and R-charts are maintained for each quality characteristic of interest.

3.2.1 Basic Principles

A variable is chosen which is characteristic of the quality of the process and can be observed or measured. This variable is regularly observed and is plotted against the time of the observation.

e.g. For some variable x, we might regularly take a sample of size n. For the $\overline{x}$-chart we plot the sample mean $\overline{x}$ while on the R-chart we plot the sample range.
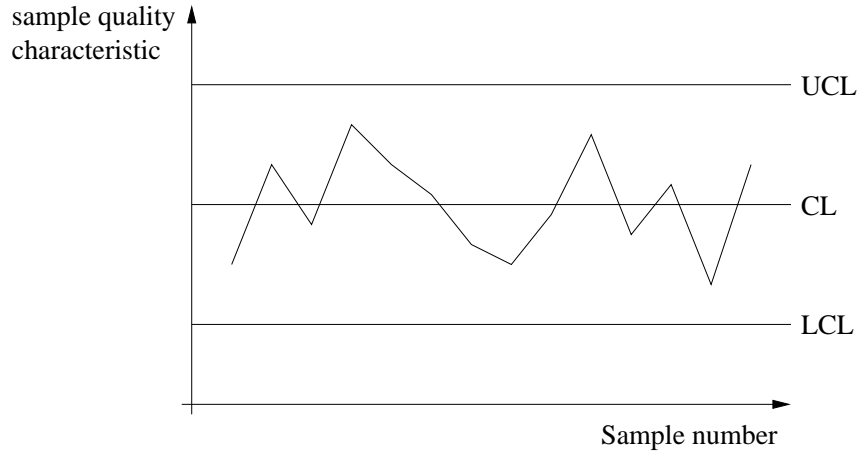
Figure 8: A typical control chart with plotted observations.

Superimposed on the plot are a number of lines:

- CENTRE LINE (CL) which represents the specified target level for the quality of the process.

- UPPER CONTROL LIMIT (UCL), LOWER CONTROL LIMIT (LCL). These are positioned so that the probability that the observed characteristic falls outside of these limits is very small when the process is in control.

  - These limits are also called the UPPER and LOWER ACTION LIMITS.

A typical control chart is shown in Figure 8. If a point falls outside these limits there is very strong evidence that assignable causes are present and action to trace and eliminate these causes should be initiated. Notice that this can be viewed as similar to falling outside a critical region in a hypothesis test and thus reject the null hypothesis, that of the process being in control.

  - The upper and lower control limits are typically drawn at ± 3 standard deviations of the centre line.
  - Note that Chebyshev's theorem states that, no matter what the true underlying distribution is, at least 89% of all observations fall within ± 3 standard deviations of the mean.
  - For a Normal distribution, over 99.73% of all observations fall within ± 3 standard deviations of the mean.

Definition 6 *(General model for a control chart)*
Let ω be a sample statistic that measures some quality characteristic of interest and suppose that the mean of ω is μ_ω and the standard deviation of ω is σ_ω. Then a control chart for ω is given by:

$$
\begin{aligned}
UCL &= \mu_\omega + L\sigma_\omega \\
CL &= \mu_\omega \\
LCL &= \mu_\omega - L\sigma_\omega
\end{aligned}
$$

where L is the "distance" of the control limits from the centre line.
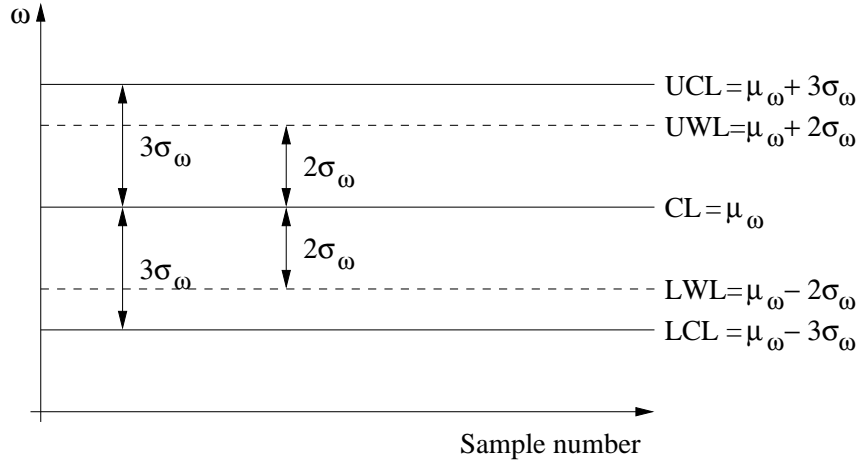
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Figure 9: A control chart with 3-sigma control limits and 2-sigma warning limits.

In order to improve sensitivity of the charts, we can add

- UPPER AND LOWER WARNING LIMITS (UWL, LWL)

  – These are usually taken at ± 2 standard deviations of the centre line.
  – Two successive points outside the warning limits is taken as good evidence that assignable causes of variation are present.

An example of such a control chart is shown in Figure 9.

3.3 Control charts for $\overline{X}$ and R: μ and σ known

Suppose a quality characteristic, X, is normally distributed with mean μ and standard deviation σ. We write $X \sim N(\mu, \sigma^2)$. If $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu, \sigma^2)$ (a random sample of observations of X) then the sample mean,

$$\overline{X} \;=\; \frac{1}{n} \sum_{i=1}^{n} X_i \sim N(\mu, \sigma^2/n).$$

With 3-sigma limits, the $\overline{x}$-chart may be constructed as

$$UCL \;=\; \mu + 3\frac{\sigma}{\sqrt{n}} \tag{2}$$

$$CL \;=\; \mu \tag{3}$$

$$LCL \;=\; \mu - 3\frac{\sigma}{\sqrt{n}} \tag{4}$$

Notice that, as a consequence of the normal assumption, we can calculate the probability of an individual sample mean falling within these limits. For any normal distribution, 99.73% of observations are within plus or minus three standard deviations of the mean. Thus, suppose we observe a sample $x_1, \ldots, x_n$ with mean

$$\overline{x} \;=\; \frac{1}{n} \sum_{i=1}^{n} x_i.$$

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If $\overline{x}$ falls outside of these limits then there is a strong indication that the observation does not come from the assumed normal distribution. In particular, that the mean may not be equal to μ. Note that we have followed the convention of using upper case letters to denote random variables (i.e. prior to being observed) and the corresponding lower case letter to denote the observed value of the random variable.

We now construct the R-chart for monitoring the variability of the process. Let

$$R \;\; = \;\; \max_i X_i - \min_i X_i \;\; = \;\; X_{\max} - X_{\min}$$

denote the range of $X_1, \ldots, X_n$. To construct the R-chart using 3-sigma limits we need to find, from Definition 6, the mean of R, $E(R)$, and the the standard deviation of R, $SD(R)$. To do this we exploit the relationship between the range of a sample of observations, $X_1, \ldots, X_n$, from a normal distribution and the standard deviation of those observations, σ.

Definition 7 *(Relative range)*
The random variable $W = R/\sigma$ is called the relative range. It has expectation $E(W) = d_2(n)$ and standard deviation $SD(W) = d_3(n)$.

Note that

- Both $E(W)$ and $SD(W)$ depend upon the size of the sample.

- The distribution of W has been extensively studied and the expectation and standard deviations tabulated, see the table handed out entitled "Appendix VI Factors for Constructing Variables Control Charts". For example, if the sample size is 10 then $E(W) = 3.078$ and $SD(W) = 0.797$ whereas if the sample size is 20 then $E(W) = 3.735$ and $SD(W) = 0.729$.

- $E(W) = d_2(n)$ is often called Hartley's constant.

We can use the mean and standard deviation of the relative range to find the corresponding mean and standard deviation of the range. As $R = \sigma W$ and σ is a constant, we find that

$$
\begin{aligned}
E(R) &\;\;=\;\; E(\sigma W) \;\;=\;\; \sigma E(W) \;\;=\;\; \sigma d_2(n), \\
SD(R) &\;\;=\;\; SD(\sigma W) \;\;=\;\; \sigma SD(W) \;\;=\;\; \sigma d_3(n).
\end{aligned}
$$

The R-chart with 3-sigma control limits is thus

$$
\begin{aligned}
UCL &\;\;=\;\; \sigma d_2(n) + 3\sigma d_3(n) & (5)\\
CL &\;\;=\;\; \sigma d_2(n) & (6)\\
LCL &\;\;=\;\; \sigma d_2(n) - 3\sigma d_3(n) & (7)
\end{aligned}
$$

The R-chart is used in parallel with the $\overline{x}$-chart. For a sample $x_1, \ldots, x_n$ we compute the observed sample range

$$r \;\; = \;\; \max_i x_i - \min_i x_i \;\; = \;\; x_{\max} - x_{\min}.$$

If r falls outside of the limits then there is evidence that the process is out of control. In particular, that the variance may not be equal to σ.

3.3.1 Control charts with non 3-sigma limits

For an $\overline{x}$-chart we can use the normal assumption to set control limits so that the probability of an observation, if the normal assumption is valid, being outside of the limits is equal to some presubscribed value α. If Z denotes the standard normal distribution, so $Z \sim N(0,1)$, then if we find the value $z_{\frac{\alpha}{2}}$ such that $P(Z < z_{\frac{\alpha}{2}}) = 1 - \frac{\alpha}{2}$ we have that

$$P\left(\mu - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

Thus,

$$
\begin{aligned}
UCL &= \mu + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \\
CL &= \mu \\
LCL &= \mu - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}
\end{aligned}
$$

is the $\overline{x}$-chart (when μ, σ known) for which there is a probability of α of being outside of the control limits when the process is in control. Notice that falling outside the UCL and LCL is equivalent a hypothesis test which rejects the hypothesis that the mean is equal to μ at the $\alpha\%$ significance level. The value $z_{\frac{\alpha}{2}} = 3$ corresponds to $\alpha = 0.0027$ (so $1 - \alpha = 0.9973$) and $P(Z < 3) = 0.99865$. If the distribution of the quality characteristic is not normally distributed then, for large n, the results will be approximately correct by an application of the Central Limit Theorem.

A similar approach can be used for the R-chart. The required values used to fix the UCL and LCL may be found from the distribution of W for a given n. Tables are often used to provide these. We comment that such a strategy is beyond the scope of this course.

3.4 $\overline{x}$ and R charts when μ, σ are unknown

The development in §3.3 allows us to set up the $\overline{x}$-chart and R-chart and use it to monitor future observations. However, in many situations μ and σ are unknown and we must first estimate them in order to set up the charts. Typically, the estimations are made from preliminary samples or subgroups taken when the process was in control. In this section we shall assume that the sample size of these preliminary samples used to estimate μ and σ is the same as that for observations once the charts have been set up. It need not be, indeed you will see an example of when the two differ in your coursework.

Suppose m samples, each containing n observations are taken on the quality characteristic. Note that typically n is small, often either 4, 5 or 6. Let $x_{j1}, \ldots, x_{jn}$ denote the n observations in the jth sample, $j = 1, \ldots, m$. The average of the jth sample is

$$\overline{x}_j = \frac{1}{n}\sum_{i=1}^{n} x_{ji}$$

and the range of this sample is

$$r_j = \max_i x_{ji} - \min_i x_{ji} = x_{j\,\max} - x_{j\,\min}.$$

The best estimate of μ is the overall mean,

$$\overline{\overline{x}} = \frac{1}{m}\sum_{j=1}^{m} \overline{x}_j.$$

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It is the best estimate in the sense that the corresponding estimator (i.e. prior to taking the observations) $\overline{\overline{X}}$ is unbiased, $E(\overline{\overline{X}}) = \mu$, and has the smallest variance.

To estimate σ we shall consider the average of our sample ranges,

$$\overline{r} = \frac{1}{m} \sum_{j=1}^{m} r_j.$$

The corresponding random variables are such that $E(R_j) = \sigma d_2(n)$ (see equation (6) for example) so that $E(\overline{R}) = \sigma d_2(n)$. Hence, $\overline{R}/d_2(n)$ is an unbiased estimator of σ. We shall use $\overline{r}/d_2(n)$ to estimate σ. Some points to note.

1. $\overline{r}/d_2(n)$ is not the only possible estimate of σ. We could use the POOLED SAMPLE STANDARD DEVIATION

$$s = \sqrt{\frac{1}{m} \sum_{j=1}^{m} s_j^2}$$

where s_j is the standard deviation of the jth sample, so that

$$s_j = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_{ji} - \overline{x}_j)^2}.$$

Traditionally, because of the small sample sizes and its simplicity, $\overline{r}/d_2(n)$ has been preferred. If n is large, then s is a better estimate as the corresponding variance is smaller.

2. If we take n_j observations in the j sample (so that the sample size depends upon the sample number) then the standard deviation of the jth sample is

$$s_j = \sqrt{\frac{1}{n_j-1} \sum_{i=1}^{n_j} (x_{ji} - \overline{x}_j)^2}$$

and the pooled standard deviation given by

$$s = \sqrt{\frac{\sum_{j=1}^{m} (n_j - 1)s_j^2}{\sum_{j=1}^{m} (n_j - 1)}}.$$

We may now use the estimates to set up the control charts. For the $\overline{x}$-chart we replace μ by its estimate $\overline{\overline{x}}$ in equations (2)- (4) and σ by its estimate $\overline{r}/r_2(n)$ in equations (2) and (4) to obtain

$$
\begin{aligned}
UCL &= \overline{\overline{x}} + 3\frac{\overline{r}}{d_2(n)\sqrt{n}} = \overline{\overline{x}} + A_2\overline{r} \\
CL &= \overline{\overline{x}} \\
LCL &= \overline{\overline{x}} - 3\frac{\overline{r}}{d_2(n)\sqrt{n}} = \overline{\overline{x}} - A_2\overline{r}
\end{aligned}
$$

where $A_2 = 3/(d_2(n)\sqrt{n})$ is as given on the hand out "Appendix VI Factors for Constructing Variables Control Charts". The R-chart is obtained by replacing σ by $\overline{r}/d_2(n)$ in equations

(5) - (7) to obtain

$$
\begin{aligned}
UCL &= \frac{\overline{r}}{d_2(n)}d_2(n) + 3\frac{\overline{r}}{d_2(n)}d_3(n) = D_4\overline{r} \\
CL &= d_2(n)\frac{\overline{r}}{d_2(n)} = \overline{r} \\
LCL &= \frac{\overline{r}}{d_2(n)}d_2(n) - 3\frac{\overline{r}}{d_2(n)}d_3(n) = D_3\overline{r}
\end{aligned}
$$

where $D_4 = 1 + 3(d_3(n)/d_2(n))$ and $D_3 = 1 - 3(d_3(n)/d_2(n))$. D_3 and D_4 are tabulated for various values of n, see the hand out "Appendix VI Factors for Constructing Variables Control Charts". Note that the range is always non-negative so that D_3 is tabulated as $\max\{0, 1 - 3(d_3(n)/d_2(n))\}$.

When using our m samples to create the estimates $\overline{\overline{x}}$ and $\overline{r}$ it is assumed that the process was in control when these samples were collected. To check this, the m sample means $\overline{x}_j$ $j = 1, \ldots, m$ should be plotted on the created $\overline{x}$-chart and the m sample ranges r_j $j = 1, \ldots, m$ plotted on the R-chart. If any out of control points are observed, the corresponding data should be discarded and fresh estimates of μ and σ obtained.

3.4.1   Example

Piston rings for an automotive engine are produced by a forging process. We wish to establish statistical control of the inside diameter of the rings manufactured by this process using $\overline{x}$ and R charts. 25 samples, each of size five, are taken when the process is in control. The inside diameter measurements are found to satisfy

$$
\sum_{j=1}^{25} \overline{x}_j = 1850.028, \ \sum_{j=1}^{25} r_j = 0.581.
$$

Construct the $\overline{x}$-chart and R-chart.

We find that

$$
\begin{aligned}
\overline{\overline{x}} &= \frac{1}{25}\sum_{j=1}^{25} \overline{x}_j = \frac{1850.028}{25} = 74.00112; \\
\overline{r} &= \frac{1}{25}\sum_{j=1}^{25} r_j = \frac{0.581}{25} = 0.02324.
\end{aligned}
$$

From the tables, for $n = 5$, $A_2 = 0.577$, $D_3 = 0$ and $D_4 = 2.115$. Thus, the $\overline{x}$-chart is

$$
\begin{aligned}
UCL &= 74.00112 + 0.577(0.02324) = 74.01453 \\
CL &= 74.00112 \\
LCL &= 74.00112 - 0.577(0.02324) = 73.98771
\end{aligned}
$$

and the R-chart is

$$
\begin{aligned}
UCL &= 2.115(0.02324) = 0.04915 \\
CL &= 0.02324 \\
LCL &= 0(0.02324) = 0.
\end{aligned}
$$

Notice that if we hadn't taken $D_3 = \max\{0, 1 - 3(d_3(5)/d_2(5))\}$ then we'd have found the $LCL = -0.00267$ but this could never be crossed as the range is always non-negative.

3.5 Interpretation of control charts

We have already commented upon INDIVIDUAL OUTLIERS. When a point falls outside the control limits, there is strong evidence that assignable causes of variation are present. Similarly, we can add warning limits to try to detect outliers, so that if two successive points fall outside the warning limits we suspect the process may be out of control.

An out of control situation can also be present even if no single point lies outside the control limits. This occurs when the pattern of plotted points on the chart exhibits non-random behaviour. We now give a number of important patterns to be aware of.

1. **Increasing or decreasing trends**. This indicates the that the process mean may have drifted. This can occur due to a gradual wearing out of tools or seasonal effects such as temperature.

2. **Shift in the process level**. Indicates that the process mean may have shifted. This could be due to the introduction of a new workforce.

3. **Cycles**. Regular repeating waves above and below the CL could indicate, for example, worker fatigue or shift pattern problems.

4. **Hugging the control limits**. Uniformly large deviations from the mean could indicate two distinct populations are being observed. For example, output from several sources (e.g. parallel machines) fed into a common stream.

5. **Hugging the centre line**. Uniformly small deviations around the mean could indicate variability has been reduced from previously expected levels and the control limits need to be tightened.

3.6 Rational subgroups

The fundamental idea in the use of control charts is the collection of sample data according to what Shewhart called the RATIONAL SUBGROUP. To illustrate this, suppose that we are using an $\overline{x}$-chart to detect changes in the process mean. The rational subgroups concept means that subgroups, or samples, should be selected so that if assignable causes are present, the chance for differences BETWEEN subgroups will be maximised, while the chance for differences due to these assignable causes WITHIN a subgroup will be minimised.

When the variation between subgroups is large relative to the variation within subgroups, we may conclude that an assignable cause is present. The within subgroup variation is our estimate of the common cause variation which some people refer to as the "short-term variation" from subgroup to subgroup.

e.g. Sampling all from the same shift.

The selection of subgroups should be made carefully as we can often make the process appear in control by stretching out the observations in the sample or by combining too much in one sample.

Number of different machines producing output into a single pool. We should ensure that we sample from a single machine, and then to different operators and so on.