Example of the transportation algorithm in action

Consider the transportation problem with costs **c**, supply **s** and demand **d** given by

$$\mathbf{c} = \begin{pmatrix} 13 & 11 & 18 & 17 \\ 2 & 14 & 10 & 1 \\ 5 & 8 & 18 & 11 \end{pmatrix}, \mathbf{s} = \begin{pmatrix} 80 \\ 100 \\ 20 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 55 \\ 70 \\ 35 \\ 40 \end{pmatrix}.$$

1. Finding an initial BFS

To solve the problem, we first find an initial basic feasible solution using the north-west corner method. We obtain T_1.

T_1	55	70	35	40	
80	55	25	0	0	
100	0	45	35	20	
20	0	0	0	20	

2. Checking if BFS is optimal

We now need to check whether this solution is optimal using the asymmetric complementary slackness conditions. For any $x_{ij} > 0$ we require $u_i + v_j = c_{ij}$. We can find the u_i, v_j by working with T_1. We first add as subscripts to each x_{ij} the corresponding cost c_{ij}. As we have one free parameter, we elect to set $u_1 = 0$. We then solve for the remaining u_i, v_j.

T_1	55	70	35	40	u_i
80	$55_{[13]}$	$25_{[11]}$	$0_{[18]}$	$0_{[17]}$	0
100	$0_{[2]}$	$45_{[14]}$	$35_{[10]}$	$20_{[1]}$	3
20	$0_{[5]}$	$0_{[8]}$	$0_{[18]}$	$20_{[11]}$	13
v_j	13	11	7	-2	$z = 2210$

Notice that we have also computed the objective function of the transportation problem

$$z = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} = 13(55) + 11(25) + 14(45) + 10(35) + 1(20) + 11(20) \ = \ 2210.$$

We need to check whether T_1 corresponds to an optimal solution. In particular, is the dual solution feasible? It will be if $u_i + v_j \leq c_{ij}$ for each i, j with $x_{ij} = 0$. We compute $t_{ij} = c_{ij} - u_i - v_j$. For example, $t_{14} = 17 - 0 - (-2) = 19$. Then, the solution will be optimal if $t_{ij} \geq 0$. We obtain

$$[t_{ij}] \ = \ \begin{bmatrix} 0 & 0 & 11 & 19 \\ -14 & 0 & 0 & 0 \\ -21 & -16 & -2 & 0 \end{bmatrix} \tag{1}$$

so that the dual solution is not feasible. Our current basic feasible solution is not optimal.

3. Moving to a new BFS

Notice that x_{31} is the "worst offender" in the sense of having the smallest value of t_{ij}. Let's introduce x_{31} into the basis by setting $x_{31} = \eta > 0$. From our initial T_1 we can perform a **perturbation loop** to do so.

T_1	55	70	35	40	
80	$55 - \eta$	$25 + \eta$	0	0	
100	0	$45 - \eta$	35	$20 + \eta$	
20	η	0	0	$20 - \eta$	

The solution is feasible for $\eta \in [0, 20]$. If $\eta = 20$ we obtain a new basic feasible solution with x_{31} replacing x_{34} in the basis. We set $\eta = 20$ and recalculate the corresponding values of u_i and v_j giving us the table T_2.

T_2	55	70	35	40	u_i
80	$35_{[13]}$	$45_{[11]}$	$0_{[18]}$	$0_{[17]}$	0
100	$0_{[2]}$	$25_{[14]}$	$35_{[10]}$	$40_{[1]}$	3
20	$20_{[5]}$	$0_{[8]}$	$0_{[18]}$	$0_{[11]}$	-8
v_j	13	11	7	-2	$z = 1790$

Notice that our new value of $z = 1790 = 2210 - 21(20)$ where our choice of $\eta = 20$ and the corresponding $t_{31} = -21$ in (1).

Cycling through the algorithm
We now check if this solution is optimal by once again calculating $t_{ij} = c_{ij} - u_i - v_j$.

$$[t_{ij}] = \begin{bmatrix} 0 & 0 & 11 & 19 \\ -14 & 0 & 0 & 0 \\ 0 & 5 & 19 & 21 \end{bmatrix} \qquad (2)$$

The dual solution is once again not feasible. We introduce x_{21} into the basis and perform a perturbation loop.

T_2	55	70	35	40	
80	$35 - \eta$	$45 + \eta$	0	0	
100	η	$25 - \eta$	35	40	
20	20	0	0	0	

The solution is feasible for $\eta \in [0, 25]$. If $\eta = 25$ we obtain a new basic feasible solution with x_{21} replacing x_{22} in the basis. We set $\eta = 25$ and recalculate the corresponding values of u_i and v_j giving us the table T_3, alongside this we compute the corresponding $[t_{ij}]$ and use this to perform, if appropriate a perturbation loop on T_3.

T_3	55	70	35	40	u_i
80	$10_{[13]} - \eta$	$70_{[11]}$	$0_{[18]} + \eta$	$0_{[17]}$	0
100	$25_{[2]} + \eta$	$0_{[14]}$	$35_{[10]} - \eta$	$40_{[1]}$	-11
20	$20_{[5]}$	$0_{[8]}$	$0_{[18]}$	$0_{[11]}$	-8
v_j	13	11	21	12	$z = 1440$

$$[t_{ij}] = \begin{bmatrix} 0 & 0 & -3 & 5 \\ 0 & 14 & 0 & 0 \\ 0 & 5 & 5 & 7 \end{bmatrix}$$

Once again, note that $\eta = 25$ and, from (2), $t_{21} = -14$ and $z = 1440 = 1790 - 14(25)$. We see that T_3 does not represent an optimal solution and we should perform a perturbation loop and, taking $\eta = 10$, introduce x_{13} into the basis for x_{11}. We obtain the table T_4.

T_4	55	70	35	40	u_i
80	$0_{[13]}$	$70_{[11]}$	$10_{[18]}$	$0_{[17]}$	0
100	$35_{[2]}$	$0_{[14]}$	$25_{[10]}$	$40_{[1]}$	-8
20	$20_{[5]}$	$0_{[8]}$	$0_{[18]}$	$0_{[11]}$	-5
v_j	10	11	18	9	$z = 1410$

$$[t_{ij}] = \begin{bmatrix} 3 & 0 & 0 & 8 \\ 0 & 11 & 0 & 0 \\ 0 & 2 & 5 & 7 \end{bmatrix}$$

Note that $\eta = 10$ with $t_{13} = -3$ and $z = 1410 = 1440 - 3(10)$. We see that this solution is optimal as all the $t_{ij} \geq 0$ so the asymmetric complementary slackness conditions are met.