MA30087/50087 - Question Sheet Seven

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2015/16 Semester I

Set: Problems Class, Thursday 19th November 2015.
Due in: Problems Class, Thursday 26th November 2015. If you are unable to make the problems class then you should ensure that you hand the work to me personally *before* the problems class is held; my office is 4W4.10.
Task: Attempt questions 1-3; questions 4-5 are additional questions which may be used for tutorial discussion.
Hint: When solving a dual problem using the simplex algorithm, you should first consider writing the dual problem as a standard maximisation problem and it is this problem you should solve.

1. Recall question 2 of Question Sheet Five. The problem was formulated as the following linear programming problem:

$$\begin{array}{rr} \text{maximise} & z = 22x_1 + 30x_2 \\ \text{subject to} & \frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 120 \\ & \frac{1}{2}x_1 + \frac{3}{4}x_2 \leq 160 \\ & x_1, x_2 \geq 0. \end{array} \tag{1}$$

   (a) Write down the corresponding dual problem to the problem (1).

   (b) Solve, using the two-phase simplex method, the dual problem. You should verify that the optimal value of the dual problem is equal to that of the original, primal problem, (1).

2. Recall the linear programming problem of question 5 of Question Sheet Four,

$$\begin{array}{rr} \text{maximise} & z = -x_1 + 2x_2 + x_3 \\ \text{subject to} & 3x_1 + x_2 - 4x_3 \leq 4 \\ & x_1 - x_2 - x_3 \leq 10 \\ & x_1 - 2x_2 + 6x_3 \leq 9 \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{2}$$

   The problem has no finite maximising solution.

   (a) Write down the corresponding dual problem to the problem (2).

   (b) By attempting to solve the dual problem using the two-phase simplex method, show that there is no feasible solution to the dual problem.
   [**Hint:** In performing the two-stage simplex method, you might find it easier to retain the column information of any artificial variable you remove from the basis. Recall that the primal simplex algorithm from Lecture 15 tells you how to identify when a linear programming problem has no feasible solution.
   Note: This result is an illustration of Corollary 1 from Lecture 17.]

3. Consider the following linear programming problem.

$$\begin{array}{ll} \text{maximise} & z = -2x_1 - 7x_2 + 2x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 \leq 1 \\ & -4x_1 - 2x_2 + 3x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{3}$$

(a) By considering the basic feasible solution of the problem (3) expressed in canonical form corresponding to the basis $\{x_1, x_3\}$, verify that the optimal solution to (3) is given by $x_1 = \frac{1}{7}$, $x_2 = 0$, $x_3 = \frac{6}{7}$.

(b) Solve, graphically, the corresponding dual problem to the problem (3).

(c) Verify that the symmetric complementary slackness conditions

$$\begin{aligned} y_i[(\mathbf{Ax})_i - b_i] &= 0 \,\forall i = 1, 2, \\ x_j[(\mathbf{A}^T\mathbf{y})_j - c_j] &= 0 \,\forall j = 1, 2, 3 \end{aligned}$$

hold for this case where $\mathbf{x} = (\frac{1}{7}, 0, \frac{6}{7})$, $\mathbf{y}$ is your optimal solution to part (b) and $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ are the corresponding matrix and vectors of the problem (3) expressed in the form: maximise $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}_3$ with $(\mathbf{Ax})_i$ denoting the ith row of $\mathbf{Ax}$ and $(\mathbf{A}^T\mathbf{y})_j$ the jth row of $\mathbf{A}^T\mathbf{y}$.

4. Consider the following linear programming problem

$$\begin{array}{ll} \text{maximise} & z = 3x_1 + 6x_2 + 2x_3 \\ \text{subject to} & 3x_1 + 4x_2 + 2x_3 \leq 200 \\ & x_1 + 3x_2 + 2x_3 \leq 100 \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{4}$$

(a) By considering the basic feasible solution of the problem (4) expressed in canonical form corresponding to the basis $\{x_1, x_2\}$, verify that the optimal solution to (4) is given by $x_1 = 40$, $x_2 = 20$, $x_3 = 0$.

(b) Write down and solve the corresponding dual problem to the problem (4).

(c) Verify that the optimal value of the dual problem is equal to that of the original, primal problem, (4).

5. Consider a linear programming problem in canonical form, maximise $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}_n$. Using the standard notation, this can be equivalently expressed as

$$\begin{array}{ll} \text{maximise} & z = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N \\ \text{subject to} & \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}_n \end{array}$$

for a basis set of variables corresponding to $\mathbf{B}$.

(a) Show that if the basic feasible solution associated with $\mathbf{B}$ is optimal then $\mathbf{c}_N^T \leq \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}$.

(b) Let $\mathbf{y}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$ and note that $\mathbf{y} \in \mathbb{R}^m$. Show that, if the basis $\mathbf{B}$ is optimal, $\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$.

(c) Show that, if the basis $\mathbf{B}$ is optimal, $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$. Hence, argue that $\mathbf{y}$ is the optimal solution to the asymmetric dual of the canonical linear programming problem.