MA30087/50087 - Question Sheet Five

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2015/16 Semester I

Set: Problems Class, Thursday 5th November 2015.
Due in: Problems Class, Thursday 12th November 2015. If you are unable to make the problems class then you should ensure that you hand the work to me personally *before* the problems class is held; my office is 4W4.10.
Task: Attempt questions 1-2; questions 3-4 are additional questions which may be used for tutorial discussion.

1. Consider the following linear programming problem.

$$\begin{array}{lr} \text{maximise} & z = 5x_1 + 6x_2 + 4x_3 \\ \text{subject to} & x_1 + x_2 + x_3 + s_1 = 10 \\ & 3x_1 + 2x_2 + 4x_3 + s_2 = 21 \\ & 3x_1 + 2x_2 + s_3 = 15 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{array} \tag{1}$$

   (a) Consider the basis $\{x_2, x_3, s_1\}$ so that $\mathbf{x}_B^T = (x_2, x_3, s_1)$ and $\mathbf{x}_N^T = (x_1, s_2, s_3)$. Using the standard notation, write the system of equations (1) in the form

$$\begin{array}{rcl} \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N & = & \mathbf{B}^{-1}\mathbf{b} \\ z - (\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N & = & \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}. \end{array} \tag{2}$$

   You may use the following matrix inverse.

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 2 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{B}^{-1} = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 1/4 & -1/4 \\ 1 & -1/4 & -1/4 \end{pmatrix}.$$

   (b) Hence, verify that $\mathbf{x}^T = (0, \frac{15}{2}, \frac{3}{2}, 1, 0, 0)$, the basic feasible solution associated to $\mathbf{B}$, is the optimal solution to the problem.

   (c) Solve the problem (1) using the simplex method. You should use a basis consisting of the slack variables as an initial basic feasible solution and determine the pivots using the $r_j\theta_j$ method. Note that your final tableau should be a representation of the equations you obtained in part (a).

2. A coffee packer blends Brazilian coffee and Colombian coffee to prepare two products, **super** and **deluxe** brands. Each kilogram of super coffee contains 0.5 kg of Brazilian coffee and 0.5 kg of Colombian coffee, whereas each kilogram of deluxe coffee contains 0.25 kg of Brazilian coffee and 0.75 kg of Colombian coffee. The packer has 120kg of Brazilian coffee and 160kg of Colombian coffee on hand. If the profit on each kilogram

of super coffee is 22 pence and the profit on each kilogram of Deluxe coffee is 30 pence, how many kilogram of each type of coffee should be blended to maximise profits? Formulate the problem as a linear programming problem and solve using the simplex method. You should use a basis consisting of the slack variables as an initial basic feasible solution.

3. Solve, using the simplex method, the following linear programming problem

$$\begin{aligned} \text{maximise} \quad & z = 3x_1 + x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 2 \\ & 2x_1 + x_2 \leq 4 \\ & -3x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{aligned}$$

4. Consider the following linear programming problem.

$$\begin{array}{llrrrrrrrrl} \text{maximise} & z = & 10x_1 & -57x_2 & -9x_3 & -24x_4 & & & & & \\ \text{subject to} & & \frac{1}{2}x_1 & -\frac{11}{2}x_2 & -\frac{5}{2}x_3 & +9x_4 & +x_5 & & & = & 0 \\ & & \frac{1}{2}x_1 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 & +x_4 & & +x_6 & & = & 0 \\ & & x_1 & & & & & & +x_7 & = & 1 \\ & & & & & & & & x_1, x_2, \ldots, x_7 & \geq & 0. \end{array}$$

When implementing the simplex method, a clear method of choosing which variables to enter and exit the basis is necessary. In lectures, we computed $r_j\theta_j$ and chose the smallest, but didn't specify a rule when there were several columns with the same score. If there is a tie in this score (for example, two choices with the same value of $r_j\theta_j$) a possible way to break the tie and to choose a variable to enter the basis is to choose the column with the largest value of r_j (the most negative element in the bottom row). If there is still a tie, we can choose between the tying variables by taking the variable with the lowest index (x_1 then x_2, then x_3 et cetera). Once we have chosen the column, if there is a tie between the rows (for example, two different rows both give the minimum for θ_j), we can break this by also choosing the one with the smallest index.

(a) Show that if this rule is used, the simplex method for the above problem cycles. (Don't forget to keep track of which variable is basic for each row. You should find that it cycles after the 7th pivot).

(b) An alternative way of selecting the correct column is **Bland's Rule** which replaces the computation of the $r_j\theta_j$ term. Instead, with Bland's rule we do the following:

- **Selecting the column:** Choose the column with the lowest index from among those with columns with negative entries in the final row;
- **Selecting the row:** Compute the ratio β_i/α_{ij} for the chosen column. If there are two rows with the same result, break the tie by choosing the variable with the lowest index.

Show that if we follow Bland's rule, we first make a different decision to (a) after the sixth pivot, and use this rule to find an optimum. *[Bland showed that if we follow these rules, then in the case of degeneracy, the Simplex algorithm will always terminate.]*