# MA30087/50087 - Question Sheet Three

Simon Shaw, s.shaw@bath.ac.uk
http://people.bath.ac.uk/masss/ma30087.html

2015/16 Semester I

**Set:** Problems Class, Thursday 22nd October 2015.
**Due in:** Problems Class, Thursday 29th October 2015. If you are unable to make the problems class then you should ensure that you hand the work to me personally *before* the problems class is held; my office is 4W4.10.
**Task:** Attempt questions 1, 2(b)(i)-(iii) and 3; questions 2(a), (b)(iv) and 4-5 are additional questions which may be used for tutorial discussion.

1. Consider the following linear programming problem.

$$\begin{array}{ll} \text{maximise} & z = 3x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{array}$$

   (a) Let $F_{(S)}$ denote the feasible region of the problem. Sketch $F_{(S)}$ and identify its extreme points. Briefly explain why $F_{(S)}$ is a closed polytope and hence why there is a finite optimal solution to the problem.

   (b) Write the problem in canonical form.

   (c) Determine all of the basic solutions of the problem in canonical form and interpret them in terms of $F_{(S)}$.

   (d) Compute the value of the objective function at each basic feasible solution. What is the optimal solution to the problem?

   (e) Let $\mathbf{x} = (1, 5/2, 1/2, 0)^T$. Use the vectors $\mathbf{x}$ and $\boldsymbol{\alpha} = (-2, 1, 1, 0)^T$, where

$$-2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

   to find a basic feasible solution to the problem in canonical form.

2. Consider a linear programming problem in canonical form with rank($\mathbf{A}$) $= m$ and feasible set $F_{(C)}$. Suppose that $\mathbf{B}$ is a $m \times m$ matrix whose columns are a linearly independent subset of the columns of $\mathbf{A}$. Without loss of generality, assume that $\mathbf{B} = (\mathbf{a}_1, \ldots, \mathbf{a}_m)$ with $\mathbf{N} = (\mathbf{a}_{m+1}, \ldots, \mathbf{a}_n)$ and $\mathbf{A} = (\mathbf{B} \,|\, \mathbf{N})$.

   (a) Suppose that $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} \in F_{(C)}$, where $\mathbf{x}_B \in \mathbb{R}^m$ and $\mathbf{x}_N \in \mathbb{R}^{n-m}$. Show that the objective function, $\mathbf{c}^T\mathbf{x}$, with $\mathbf{c} = \begin{pmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{pmatrix}$ where $\mathbf{c}_B \in \mathbb{R}^m$ and $\mathbf{c}_N \in \mathbb{R}^{n-m}$, can be expressed as

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \left(\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}\right)\mathbf{x}_N. \quad (1)$$

(b) Consider the following linear programming problem.

$$\begin{array}{ll} \text{maximise} & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}_5 \end{array}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T$,

$$\mathbf{A} = \begin{pmatrix} -10 & -5 & 1 & 0 & 0 \\ -2 & -3 & 0 & 1 & 0 \\ -2 & -8 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} -10 \\ -3 \\ -4 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} -40 \\ -80 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let $\mathbf{a}_j$ denote the $j$th column of $\mathbf{A}$ for $j = 1, \ldots, 5$. Consider the following choices of the matrix $\mathbf{B}$.

i. $\mathbf{B} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$
ii. $\mathbf{B} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5)$
iii. $\mathbf{B} = (\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4)$
iv. $\mathbf{B} = (\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5)$

In each case, express $\mathbf{c}^T\mathbf{x}$ in the form given by (1) and use this to determine whether or not the basic feasible solution corresponding to $\mathbf{B}$ is optimal.
[**Hint:** this is the cost minimisation problem from Lecture 2 written in canonical form; we found the basic solutions for this in question 4. of Question Sheet Two. Solution Sheet Two contains all the relevant inverses of $\mathbf{B}$ that you need.]

3. Suppose that the $m \times n$ matrix $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ has rank $m$ and that for some $k < m$, $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are linearly independent. Show that $m - k$ vectors from the remaining $n - k$ vectors can be adjoined to $\mathbf{a}_1, \ldots, \mathbf{a}_k$ to form a set of $m$ linearly independent vectors.

4. Suppose the canonical form of a linear programming problem is given by the constraint matrix $\mathbf{A}$ and vector $\mathbf{b}$, where:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 0 & 0 & 3 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

A feasible solution to the problem is $\mathbf{x} = (1/2, 1, 3/2, 1, 1/2)^T$.

(a) Use the vector $\boldsymbol{\alpha} = (0, -1/2, 5/6, 1, 2/3)^T$ to show that $\mathbf{x}$ is not an extreme point of the feasible set of the problem.

(b) Use the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = (-4, 1, -3, -2, 0)^T$ to find a basic feasible solution to the problem.

5. Show that introducing slack variables does not change the number of extreme points of the feasible set by proving that $\mathbf{x} \in \mathbb{R}^n$ is an extreme point of $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}_n\}$ if and only if $\mathbf{x}' = \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix} \in \mathbb{R}^{n+m}$ is an extreme point of $\{\mathbf{x}' \in \mathbb{R}^{n+m} : \mathbf{A}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}_n, \mathbf{s} \geq \mathbf{0}_m\}$.