MA30087/50087 - Question Sheet One

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2015/16 Semester I

Set: Problems Class, Thursday 8th October 2015.
Due in: Problems Class, Thursday 15th October 2015. If you are unable to make the problems class then you should ensure that you hand the work to me personally *before* the problems class is held; my office is 4W4.10.
Task: Attempt questions 1-3; questions 4-5 are additional questions and may be used for tutorial discussion.

1. Using the graphical method, solve the following linear programming problems.

   (a)

$$\begin{array}{rr} \text{maximise} & z = x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 10 \\ & 2x_1 + x_2 \leq 16 \\ & -x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{array}$$

   (b)

$$\begin{array}{rr} \text{minimise} & z = x_1 + 3x_2 \\ \text{subject to} & x_1 + 2x_2 \geq 6 \\ & x_1 - x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{array}$$

   (c) Minimise $2x_1 - x_2$ subject to the constraints given in part (b).

2. Express the following linear programming problems as canonical problems.

   (a)

$$\begin{array}{rr} \text{minimise} & z = 2x_1 + x_2 + 2x_3 \\ \text{subject to} & 1 \leq x_1 + x_2 \leq 6 \\ & 2 \leq x_1 + x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

   (b)

$$\begin{array}{rr} \text{minimise} & z = x_1 + 4x_2 + x_3 \\ \text{subject to} & 2x_1 + x_2 + 3x_3 = 8 \\ & x_1 \geq -1, x_3 \geq 1. \end{array}$$

3. By first showing that if S_1 and S_2 are convex sets then $S_1 \cap S_2$ is a convex set, prove that the intersection of any collection of convex sets is convex.

4. (a) Solve the following linear programming problem graphically.

$$\begin{array}{ll} \text{minimise} & z = x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \geq 3 \\ & x_1 + 4x_2 \geq 6 \\ & 3x_1 + 2x_2 \geq 7 \\ & x_1, x_2 \geq 0. \end{array}$$

   (b) Consider the problem of maximising $ax_1 + bx_2$, where a and b are non-zero real numbers, subject to the constraints given in part (a). Find values of a and b such that:

   i. There is a unique optimal solution.
   ii. There are infinitely many optimal solutions.
   iii. There are no optimal solutions.

5. Consider a linear programming problem given in standard form (S). That is to say, we seek a vector $\mathbf{x} \in \mathbb{R}^n$ that will

$$\begin{array}{ll} \text{maximise} & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}_n \end{array}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$. Write down the associated problem in canonical form (C). Prove that if (C) has an optimal solution then its restriction to $\mathbb{R}^n$ is also the optimal solution to (S). Conversely suppose that (S) has an optimal solution $\mathbf{x} \in \mathbb{R}^n$. Prove that there exists an optimal solution to (C) whose restriction to $\mathbb{R}^n$ is $\mathbf{x}$.