

Figure 1: The region, shaded, for which all the feasible solutions to Example 1 must lie. The region has four extreme points: $(0, 2)$, $(0.75, 0.5)$, $(1.2, 0.2)$ and $(2, 0)$. The lines $x_1 = 0$ (for $x_2 \geq 2$) and $x_2 = 0$ (for $x_1 \geq 2$) are **extreme rays**. This feasible set is unbounded.

Example 1 (*Cost minimisation problem*) We wish to feed cattle, meeting nutritional requirements at minimum cost. A farmer has two feeds “Mootastic” and “Udderly Wonderful” at her disposal. There are certain nutritional requirements which stipulate that cattle should receive minimum quantities of carbohydrate, vitamins and protein. The table below gives the number of units per kilo of the nutrients required along with the cost of the two feeds.

Food	Carbo.	Vit.	Pro.	Cost (pence/kilo)
Mootastic	10	2	2	40
Udderly Wonderful	5	3	8	80
Daily Requirement	10	3	4	

Let x_1 denote the number of kilos of Mootastic and x_2 the number of kilos of Udderly Wonderful used per day. Our objective is to:

$$\begin{aligned}
 &\text{minimise} && z = 40x_1 + 80x_2 \\
 &\text{subject to} && 10x_1 + 5x_2 \geq 10 \\
 &&& 2x_1 + 3x_2 \geq 3 \\
 &&& 2x_1 + 8x_2 \geq 4 \\
 &&& x_1, x_2 \geq 0.
 \end{aligned}$$

This is a linear programming problem with two variables and three constraints. As this is a problem in \mathbb{R}^2 we can solve the problem graphically. In Figure 1 we illustrate the feasible region, that is the set of points with coordinates (x_1, x_2) that satisfy all of the constraints. The objective is to minimise the linear function $z = 40x_1 + 80x_2$. Figure 2 illustrates how we do this. If we fix a value of z then $40x_1 + 80x_2 \leq z$ corresponds to points below the line. Decreasing the value of z corresponds to translating the objective function through the feasible region in a southwest direction. For example, Figure 2 shows the objective

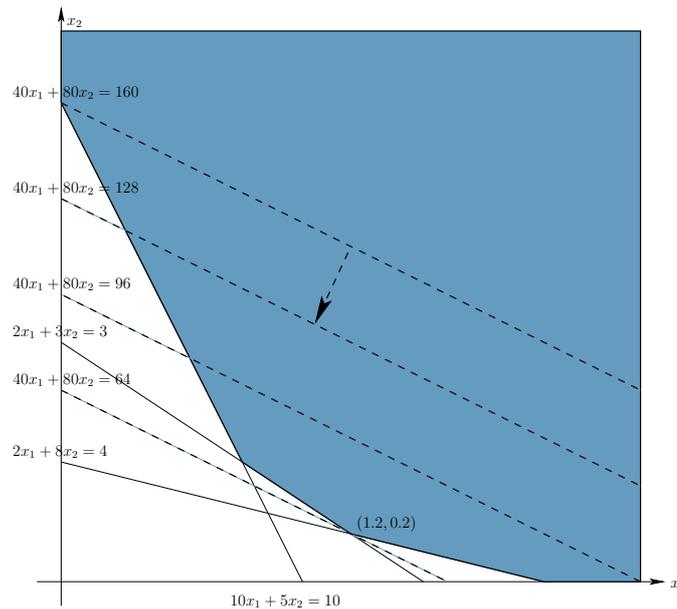


Figure 2: Translating the objective function of Example 1 through the feasible region. The minimum of 64 occurs at the extreme point $(1.2, 0.2)$.

function for decreasing values of z . Note that when $z = 64$ we obtain the minimum at the extreme point $(1.2, 0.2)$ where $z = 64$. This extreme point corresponds to $2x_1 + 3x_2 = 3$ and $2x_1 + 8x_2 = 4$: the constraints $2x_1 + 3x_2 \geq 3$ and $2x_1 + 8x_2 \geq 4$ are **active** or **binding** or **tight** at this point. Note that whilst the feasible region is unbounded, there was a unique solution to the problem.¹

Example 2 Consider the linear programming problem:

$$\begin{aligned}
 &\text{maximise} && z = 18x_1 + 6x_2 \\
 &\text{subject to} && 3x_1 + x_2 \leq 120 \\
 &&& x_1 + 2x_2 \leq 160 \\
 &&& x_1 \leq 35 \\
 &&& x_1, x_2 \geq 0.
 \end{aligned}$$

Once again this is a problem with two variables and three constraints. Consequently, we can solve this \mathbb{R}^2 problem graphically. In Figure 3 we illustrate the feasible region, that is the set of points with coordinates (x_1, x_2) that satisfy all of the constraints. The objective is to maximise the linear function $z = 18x_1 + 6x_2$. Figure 4 illustrates how we do this. If we fix a value of z then $18x_1 + 6x_2 \geq z$ corresponds to points above the line. Increasing the value of z corresponds to translating the objective function through the feasible region in a northeast direction. For example, Figure 4 shows the objective function for increasing values of z . Note that when $z = 720$ we obtain the maximum but there are an infinite number of points where this is achieved. These do include two extreme points: the point $(16, 72)$, where the constraints $x_1 + 2x_2 \leq 160$ and $3x_1 + x_2 \leq 120$ are active, and the point $(35, 15)$, where the constraints $x_1 \leq 35$ and $3x_1 + x_2 \leq 120$ are active.

¹Note also that had we sought to maximise $z = 40x_1 + 80x_2$ over the feasible region we would have found that there was no optimal solution.

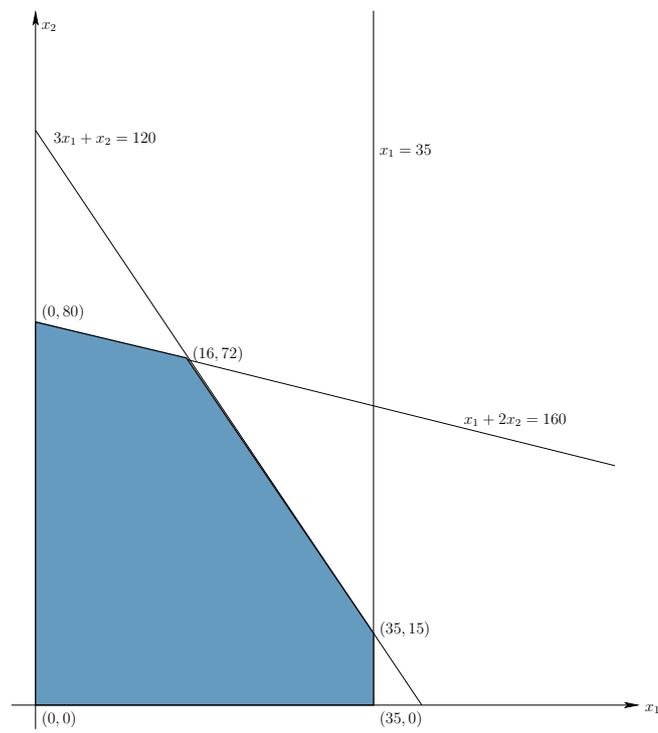


Figure 3: The region, shaded, for which all the feasible solutions to Example 2 must lie. The region has five extreme points: $(0, 0)$, $(0, 80)$, $(16, 72)$, $(35, 15)$ and $(35, 0)$. This feasible set is bounded.

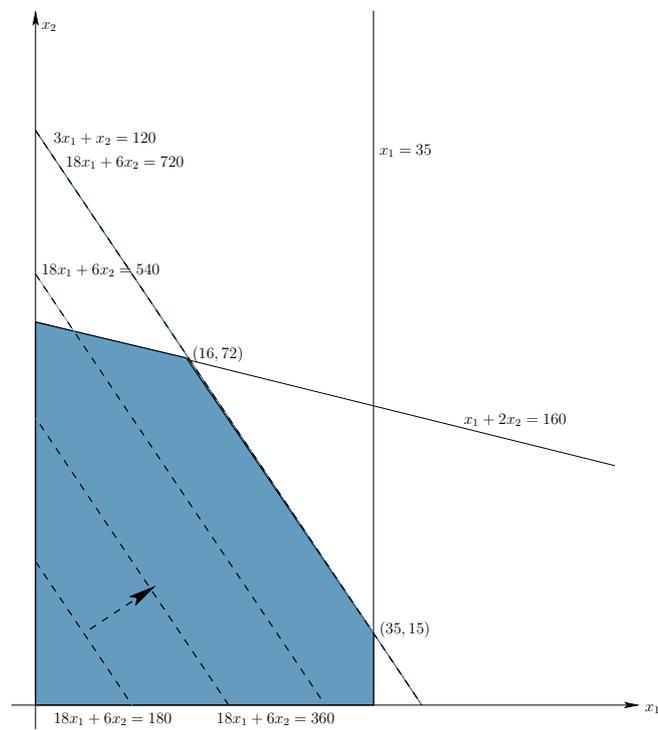


Figure 4: Translating the objective function of Example 2 through the feasible region. The maximum of 720 occurs is not unique, but does occur at the extreme points $(16, 72)$ and $(35, 15)$.