26/11/2015.

PROBLEMS CLASS: 3WN2.1.

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5. CS conditions for problem in canonical form:

$\underline{x} \in F_{(P)}, \underline{y} \in F_{(D)}$ optimal $\Leftrightarrow$ $\underline{x} \in F_{(P)}, \underline{y} \in F_{(D)}$ and

$$x_j \left[(\underline{A}^T \underline{y})_j - c_j \right] = 0 \ \forall j = 1, \dots, n.$$

CONVERT TO A MAXIMISATION PROBLEM:

maximise $z_1 = -x_1 - 6x_2 - 2x_3 + x_4 - x_5 + 3x_6$

subject to

$$x_1 + 2x_2 + x_3 + 5x_6 = 3$$
$$-3x_2 + 2x_3 + x_4 + x_6 = 1$$
$$5x_2 + 3x_3 + x_5 - 2x_6 = 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

i.e. $\underline{A} = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 5 \\ 0 & -3 & 2 & 1 & 0 & 1 \\ 0 & 5 & 3 & 0 & 1 & -2 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

DUAL IS minimise $z' = 3y_1 + y_2 + 2y_3$

subject to $\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} -1 \\ -6 \\ -2 \\ 1 \\ -1 \\ 3 \end{pmatrix}$

PROPOSED SOLUTION: $\underline{x} = \left(0, \frac{16}{29}, 0, \frac{66}{29}, 0, \frac{11}{29}\right)$ GIVES:

$$\underline{c}^T \underline{x} = -6\left(\frac{16}{29}\right) + \frac{66}{29} + 3\left(\frac{11}{29}\right) = \frac{3}{29}.$$

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NOTE: CAN CHECK THAT $\underline{x}$ IS FEASIBLE

i.e. $\underline{A}\,\underline{x} = \underline{b}$ and $\underline{x} \geq \underline{0}_6$.

ASYMMETRIC CS CONDITIONS ARE EQUIVALENT TO:

$x_j > 0 \Rightarrow (\underline{A}^T \underline{y})_j = c_j$ CONSTRAINT IS TIGHT

$(\underline{A}^T \underline{y})_j > c_j \Rightarrow x_j = 0$ (D) CONSTRAINT IS SLACK $\Rightarrow$ (P) VARIABLE 0.

NOTE: WE HAVE $x_2 > 0$, $x_4 > 0$ and $x_6 > 0$ SO CS CONDITIONS REQUIRE

$$\begin{aligned} (\underline{A}^T \underline{y})_2 &= c_2 & 2y_1 - 3y_2 + 5y_3 &= -6 \\ (\underline{A}^T \underline{y})_4 &= c_4 & y_2 &= 1 \\ (\underline{A}^T \underline{y})_6 &= c_6 & 5y_1 + y_2 - 2y_3 &= 3 \end{aligned}$$

[NB. BFS: m IN BASIS, $n-m$ NON-BASIC. AT LEAST $n-m$ ZERO VARIABLES AND AT MOST m NON-ZERO VARIABLES. $\underline{y} \in \mathbb{R}^m$ SO $(\underline{A}^T \underline{y})_j = c_j$ m EQUATIONS FOR m UNKNOWNS.]

AS $y_2 = 1$ EQUATIONS REDUCE TO: $\left.\begin{aligned} 2y_1 + 5y_3 &= -3 \\ 5y_1 - 2y_3 &= 2 \end{aligned}\right\}$ $y_1 = \frac{4}{29}$

$y_3 = \frac{-19}{29}$

PROPOSED SOLUTION $\left(\frac{4}{29}, 1, -\frac{19}{29}\right)$ IS IT FEASIBLE?

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & -2 \end{pmatrix} \begin{pmatrix} 4/29 \\ 1 \\ -19/29 \end{pmatrix} = \begin{pmatrix} 4/29 \\ -6 \\ 5/29 \\ 1 \\ -19/29 \\ 3 \end{pmatrix} \begin{matrix} > -1 & x_1 = 0 \\ \text{TIGHT} & \\ > -2 & x_3 = 0 \\ \text{TIGHT} & \\ > -1 & x_5 = 0 \\ \text{TIGHT} & \end{matrix}$$

$\Rightarrow$ $\underline{y}$ is feasible, $\underline{x}$ is feasible and CS conditions satisfied.

$\Rightarrow$ $\underline{x}, \underline{y}$ optimal.

NB. $\underline{b}^T \underline{y} = 3\left(\frac{4}{29}\right) + 1 + 2\left(\frac{-19}{29}\right) = \frac{3}{29}$.

WHAT IS THE DUAL SOLUTION?

FOR (P), $z = \underline{c}_B^T B^{-1} \underline{b} + (\underline{c}_N^T - \underline{c}_B^T B^{-1} \underline{N}) \underline{x}_N$.

IF $\underline{x}$ IS THE BFS CORRESPONDING TO BASIS $\underline{B}$ THEN

$$z = \underline{c}_B^T B^{-1} \underline{b} \quad \text{AND} \quad \underline{c}_N^T \le \underline{c}_B^T B^{-1} \underline{N}.$$

IF $\underline{x}$ IS OPTIMAL.

i.e. $z = \underline{c}_B^T B^{-1} \underline{b}$ IS OPTIMAL VALUE OF z.

WEAK DUALITY $\Rightarrow$ $\underline{c}_B^T B^{-1} \underline{b} = \underline{b}^T \underline{y}$ i.e. $\underline{y}^T = \underline{c}_B^T B^{-1}$.

NOTE: $\underline{y}^T \underline{A} = \underline{y}^T (\underline{B} \mid \underline{N}) = \underline{c}_B^T B^{-1} (\underline{B} \mid \underline{N})$

$$= (\underline{c}_B^T \mid \underline{c}_B^T B^{-1} \underline{N})$$
$$\ge (\underline{c}_B^T \mid \underline{c}_N^T)$$

i.e. $\underline{y}^T \underline{A} \ge \underline{c}^T$

$\Rightarrow \underline{A}^T \underline{y} \ge \underline{c}$.

$$\underline{B} = \begin{pmatrix} 2 & 0 & 5 \\ -3 & 1 & 1 \\ 5 & 0 & -2 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 2/29 & 0 & 5/29 \\ 11/29 & 1 & 17/29 \\ 5/29 & 0 & -2/29 \end{pmatrix} \quad \underline{c}_B = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$$

$$\underline{y}^T = \underline{c}_B^T B^{-1}.$$