19/11/2015 PROBLEMS CLASS: 3WN 2.1.

(A).

4.
$$\text{maximum } z = 3x_1 + 6x_2 + 2x_3$$
$$\text{subject to } 3x_1 + 4x_2 + 2x_3 \le 200$$
$$x_1 + 3x_2 + 2x_3 \le 100$$
$$x_1, x_2, x_3 \ge 0$$

DUAL PROBLEM IS minimum $\underline{b}^T \underline{y}$
subject to $\underline{A}^T \underline{y} \ge \underline{c}$
$\underline{y} \ge \underline{0}_2$

$\underline{b}^T = (200 \;\; 100)$

$\underline{A} = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 3 & 2 \end{pmatrix}$

$\underline{c}^T = (3 \;\; 6 \;\; 2)$

i.e. minimum $200y_1 + 100y_2$
subject to
$$3y_1 + y_2 \ge 3$$
$$4y_1 + 3y_2 \ge 6$$
$$2y_1 + 2y_2 \ge 2$$
$$y_1, y_2 \ge 0$$

(a). IN CANONICAL FORM WE HAVE:

$$\text{maximum } z = 3x_1 + 6x_2 + 2x_3$$
$$\text{subject to } 3x_1 + 4x_2 + 2x_3 + s_1 = 200$$
$$x_1 + 3x_2 + 2x_3 + s_2 = 100$$
$$x_1, x_2, x_3, s_1, s_2 \ge 0$$

DUAL PROBLEM IS minimum $\underline{b}^T \underline{y}$
subject to $\underline{A}^T \underline{y} \ge \underline{c}$

$\underline{b}^T = (200 \;\; 100)$

$\underline{A} = \begin{pmatrix} 3 & 4 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 & 1 \end{pmatrix}$

$\underline{c}^T = (3 \;\; 6 \;\; 2 \;\; 0 \;\; 0)$

i.e. minimum $200y_1 + 100y_2$
subject to
$$3y_1 + y_2 \ge 3$$
$$4y_1 + 3y_2 \ge 6$$
$$2y_1 + 2y_2 \ge 2$$
$$\left.\begin{aligned} y_1 &\ge 0 \\ y_2 &\ge 0 \end{aligned}\right\} \text{SLACK VARIABLES RESTORE NON-NEGATIVITY OF THE DUAL VARIABLES.}$$

B

$$\underset{\sim}{B} = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}, \quad \underset{\sim}{B}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}, \quad \underset{\sim}{B}^{-1}\underset{\sim}{N} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2/5 & 3/5 & -4/5 \\ 4/5 & -1/5 & 3/5 \end{pmatrix}$$

$$\underset{\sim}{B}^{-1}\underset{\sim}{b} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 40 \\ 20 \end{pmatrix}.$$

$$\underset{\sim}{c}_N^T - \underset{\sim}{c}_B^T \underset{\sim}{B}^{-1}\underset{\sim}{N} = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 6 \end{pmatrix}\begin{pmatrix} -2/5 & 3/5 & -4/5 \\ 4/5 & -1/5 & 3/5 \end{pmatrix}$$

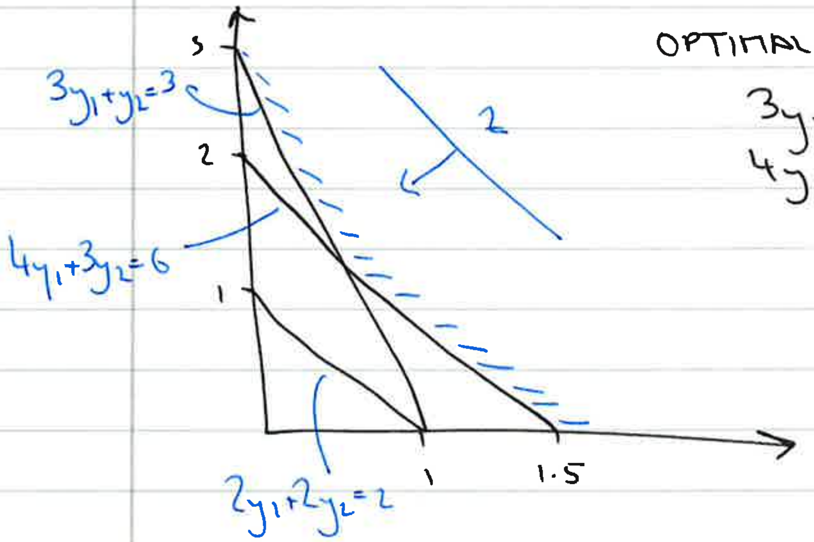
$$= \begin{pmatrix} 2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 18/5 & 3/5 & 6/5 \end{pmatrix}$$

$$= \begin{pmatrix} -8/5 & -3/5 & -6/5 \end{pmatrix}$$

ALL THE REDUCED COSTS ARE NEGATIVE, THIS IS THE OPTIMAL SOLUTION.

$$z = \underset{\sim}{c}_B^T \underset{\sim}{B}^{-1}\underset{\sim}{b} = \begin{pmatrix} 3 & 6 \end{pmatrix}\begin{pmatrix} 40 \\ 20 \end{pmatrix} = \underline{\underline{240}}.$$

(b) SOLVING THE DUAL PROBLEM IS TWO-DIMENSIONAL.



OPTIMAL POINT IS

$$\left.\begin{aligned} 3y_1 + y_2 &= 3 \\ 4y_1 + 3y_2 &= 6 \end{aligned}\right\} \quad \begin{aligned} 5y_1 &= 3 \\ y_1 &= 3/5 \\ y_2 &= 6/5. \end{aligned}$$

NOTE: $200\left(\frac{3}{5}\right) + 100\left(\frac{6}{5}\right)$

$= \underline{\underline{240}}$

NOTE: DUAL SOLUTION OCCURS WITH:

$$3y_1 + y_2 = 2 \qquad (\underline{A}^T \underline{y})_1 - c_1 = 0$$

$$4y_1 + 3y_2 = 6 \qquad (\underline{A}^T \underline{y})_2 - c_2 = 0$$

$$2y_1 + 2y_2 > 2 \qquad (\underline{A}^T \underline{y})_3 - c_3 \neq 0$$

BUT $x_3 \neq 0 \Rightarrow x_j [(\underline{A}^T \underline{y})_j - c_j] = 0 \ \forall j = 1,2,3$

EQUALLY, $y_i [(\underline{A}\,\underline{x})_i - b_i] = 0 \ \forall \ i = 1,2.$

COMPLEMENTARY SLACKNESS CONDITIONS.

NOTE $\underline{y}^T = \underline{c}_B^T \underline{B}^{-1} = (3 \ \ 6) \frac{1}{5} \begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix} = (3/5 \ \ 6/5)$

5. $$z = \underline{c}_B^T \underline{B}^{-1} \underline{b} + (\underline{c}_N^T - \underline{c}_B^T \underline{B}^{-1} \underline{N}) \underline{x}_N.$$

a). If $\underline{x}^*(\underline{B})$ is the bfs associated to $\underline{B}$ then $z(\underline{x}^*) = \underline{c}_B^T \underline{B}^{-1} \underline{b}$

This is optimal if

$$\underline{c}_N^T - \underline{c}_B^T \underline{B}^{-1} \underline{N} \leq \underline{0}^T_{n-m}$$

i.e. $$\underline{c}_N^T \leq \underline{c}_B^T \underline{B}^{-1} \underline{N}.$$

b). $\underline{y}^T = \underline{c}_B^T \underline{B}^{-1}$ $\qquad \underline{y}^T \underline{A} = \underline{y}^T (\underline{B} | \underline{N}) = (\underline{y}^T \underline{B} \ \ \underline{y}^T \underline{N})$

$$= (\underline{c}_B^T \ \ \underline{c}_B^T \underline{B}^{-1} \underline{N})$$

$$\geq (\underline{c}_B^T \ \ \underline{c}_N^T) = \underline{c}^T$$

i.e. $\underline{y}^T \underline{A} \geq \underline{c}^T$

i.e. $\underline{A}^T \underline{y} \geq \underline{c}.$

$\underline{y}$ IS FEASIBLE FOR THE DUAL.

c). $z' = \underline{y}^T \underline{b} = \underline{c}_B^T \underline{B}^{-1} \underline{b} = z(\underline{x}^*)$.

OBJECTIVE FUNCTIONS FOR PRIMAL AND DUAL EQUAL. THUS, $\underline{y}$ OPTIMAL FOR (D), $\underline{x}^*$ OPTIMAL FOR (P).

IF YOU DON'T BELIEVE THIS, WE'LL PROVE IT IN THE PROBLEMS CLASS NEXT WEEK.