12|11|2015. PROBLEMS CLASS: 3WN2.1. Ⓐ.

4. maximise $z = 2x_1 + x_2 + 3x_3$

$$x_1 - 3x_2 + \tfrac{1}{2}x_3 = 6$$
$$-x_1 + 8x_2 + \tfrac{1}{2}x_3 \leq 4$$
$$-3x_1 + 4x_2 + \tfrac{1}{2}x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0.$$

①. ADD IN SLACK VARIABLES (i.e. CONVERT TO CANONICAL FORM).

$$\underline{A}\,\underline{x} = \underline{b} \quad \begin{cases} x_1 - 3x_2 + \tfrac{1}{2}x_3 = 6 \\ -x_1 + 8x_2 + \tfrac{1}{2}x_3 + s_1 = 4 \\ -3x_1 + 4x_2 + \tfrac{1}{2}x_3 - s_2 = 1 \end{cases} \qquad \underline{A} = \begin{pmatrix} 1 & -3 & \tfrac{1}{2} & 0 & 0 \\ -1 & 8 & \tfrac{1}{2} & 1 & 0 \\ -3 & 4 & \tfrac{1}{2} & 0 & -1 \end{pmatrix}$$

$\underline{x} \geq \underline{0}_5$ $\quad x_1, x_2, x_3, s_1, s_2 \geq 0$ NO IMMEDIATE BFS.

②. ADD IN ARTIFICIAL VARIABLES.

$$\underline{A}'\,\underline{x}' = \underline{b} \quad \begin{cases} x_1 - 3x_2 + \tfrac{1}{2}x_3 + u_1 = 6 \\ -x_1 + 8x_2 + \tfrac{1}{2}x_3 + s_1 = 4 \\ -3x_1 + 4x_2 + \tfrac{1}{2}x_3 - s_2 + u_2 = 1 \end{cases} \qquad \underline{A}' = \begin{pmatrix} 1 & -3 & \tfrac{1}{2} & 0 & 0 & 1 & 0 \\ -1 & 8 & \tfrac{1}{2} & 1 & 0 & 0 & 0 \\ -3 & 4 & \tfrac{1}{2} & 0 & -1 & 0 & 1 \end{pmatrix}$$

$\underline{x}' \geq \underline{0}_7$ $\quad x_1, x_2, x_3, s_1, s_2, u_1, u_2 \geq 0$

INITIAL BASIS $\{u_1, s_1, u_2\}$ WITH $u_1 = 6, s_1 = 4, u_2 = 1$.

PHASE I. AUXILIARY PROBLEM

maximise $z' = -u_1 - u_2$

subject to $\underline{A}'\underline{x}' = \underline{b}$

$\underline{x}' \geq \underline{0}_7$.

NOW $z' \leq 0$ AND $z' = 0$ ACHIEVED WITH $u_1 = u_2 = 0$.
ANY SOLUTION WITH $u_1 = u_2 = 0$ WILL HAVE $\underline{A}\,\underline{x} = \underline{b}$.
IF $u_1 = u_2 = 0$ NON-BASIC THEN BFS OF $\underline{A}\,\underline{x} = \underline{b}$
(VIA SIMPLEX ALGORITHM).

(B).

USE THIS BFS AS AN INITIAL BFS FOR:

PHASE II

$$\begin{aligned} \text{maximise} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & A \underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{aligned}$$

WE NEED TO EXPRESS $z' = -u_1 - u_2$ AS A FUNCTION OF THE NON-BASIC VARIABLES.

$$\begin{aligned} x_1 - 3x_2 + \tfrac{1}{2}x_3 + u_1 &= 6 \\ -3x_1 + 4x_2 + \tfrac{1}{2}x_3 - s_2 + u_2 &= 1 \end{aligned}$$

SO, $$-2x_1 + x_2 + x_3 - s_2 + u_1 + u_2 = 7$$

$$\Rightarrow \quad -u_1 - u_2 = -7 - 2x_1 + x_2 + x_3 - s_2$$

OR, $$z' + 2x_1 - x_2 - x_3 + s_2 = -7.$$

** HANDOUT **

5. COLUMN OF x_2 SUGGESTS THAT z' IS UNBOUNDED.

$$z' = -6 + 5x_2 - 2x_3 + s_1$$

BUT $x_2 = M$ FEASIBLE $\forall M > 0 \Rightarrow z' \to \infty$ AS $M \to \infty$.
CAN'T HAPPEN AS, BY DESIGN, $\underline{\underline{z' \leq 0}}$.

QUESTION SHEET FIVE (REVISITED)

3.

T_1	z	x_1	x_2	s_1	s_2	s_3	
s_1	0	1 [2]	−1	1	0	0	2
s_2	0	2 [2]	1 [4]	0	1	0	4
s_3	0	−3	2 [3]	0	0	1	6
	1	−3	−1	0	0	0	0

(IN THE PROBLEMS CLASS LAST WEEK WE SOLVED THIS GRAPHICALLY. THE OPTIMAL SOLUTION IS $z=6$ AT $(2,0,0,0,12)$.)

$z = 0 + 3x_1 + x_2$. CONSIDER ADDING EITHER x_1 OR x_2 TO THE BASIS. TAKING $x_1 = 2$ WILL ADD 6 TO z

TAKING $x_2 = 3$ WILL ADD 3 TO z.

NOTE IF $x_1 = 2$ THEN BOTH s_1 AND s_2 GO TO ZERO. WE'LL GET A DEGENERATE SOLUTION.

WANT TO MAINTAIN A BASIS OF SIZE $m=3$ SO HAVE TO DECIDE WHICH OF s_1, s_2 TO PUSH OUT. LET'S PUSH OUT s_1.

T_2	z	x_1	x_2	s_1	s_2	s_3	
x_1	0	1	−1	1	0	0	2
s_2	0	0	3 [0]	−2	1	0	0
s_3	0	0	−1	3	0	1	12
	1	0	−4	3	0	0	6

BASIC.

THIS IS THE POINT $(2,0,0,\boxed{0},12)$ WITH $z=6$. WE KNOW THIS IS OPTIMAL. HOWEVER UNDER THIS BASIS $z = 6 + 4x_2 - 3s_1$ SO WE CAN'T CONFIRM IT.

LET'S ADD x_2 TO THE BASIS, PUSHING OUT s_2.

D

T_3	z	x_1	x_2	s_1	s_2	s_3	
x_1	0	1	0	$1/3$	$1/3$	0	2
x_2	0	0	1	$-2/3$	$1/3$	0	0
s_3	0	0	0	$7/3$	$1/3$	1	12
	1	0	0	$1/3$	$4/3$	0	6

BFS IS $(2, \underset{\text{BASIC}}{\boxed{0}}, 0, 0, 12)$ WITH $z = 6 - \frac{1}{3}s_1 - \frac{4}{3}s_2 \leq 6$

I CAN NOW IDENTIFY THIS AS THE OPTIMAL SOLUTION.

DEGENERATE SOLUTIONS CAN POSE PROBLEMS FOR THE SIMPLEX ALGORITHM AND IF YOU'RE NOT CAREFUL YOU CAN CYCLE (i.e. RETURN TO SAME SOLUTION)

THERE'S AN EXAMPLE OF THIS ON QUESTION 4.

Example of the two-phase artificial variables method; question 4 of Question Sheet Six

We begin by adding in relevant slack variables:

(*)

$$\begin{array}{ll} \text{maximise} & 2x_1 + x_2 + 3x_3 \\ \text{subject to} & x_1 - 3x_2 + \frac{1}{2}x_3 = 6 \\ & -x_1 + 8x_2 + \frac{1}{2}x_3 + s_1 = 4 \\ & -3x_1 + 4x_2 + \frac{1}{2}x_3 - s_2 = 1 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0. \end{array}$$

Then the first and third equations also require an artificial variable, and our first phase is to maximise $z' = -u_1 - u_2$.

T_1	z'	z	x_1	x_2	x_3	s_1	s_2	u_1	u_2	
u_1	0	0	1	-3	$\frac{1}{2}_{[12]}$	0	0	1	0	6
s_1	0	0	-1	$8_{[\frac{1}{2}]}$	$\frac{1}{2}_{[8]}$	1	0	0	0	4
u_2	0	0	-3	$4_{[\frac{1}{4}]}$	$\frac{1}{2}_{[2]}$	0	-1	0	1	1
II	0	1	-2	-1	-3	0	0	0	0	0
I	1	0	2	-1	-1	0	1	0	0	-7

$z' = -7 - 2x_1 + x_2 + x_3 - s_2$. Introduce either x_2 or x_3 into the basis. Adding x_2 adds 1/4 to z' whilst adding x_3 adds 2 to z'. Add x_3 pushing out u_2.

We should introduce x_3 for u_2. The new tableau is:

T_2	z'	z	x_1	x_2	x_3	s_1	s_2	u_1	u_2	
u_1	0	0	$4_{[\frac{5}{4}]}$	-7	0	0	$1_{[5]}$	1	–	5
s_1	0	0	$2_{[\frac{3}{2}]}$	4	0	1	$1_{[3]}$	0	–	3
x_3	0	0	-6	8	1	0	-2	0	–	2
II	0	1	-20	23	0	0	-6	0	–	6
I	1	0	-4	7	0	0	-1	0	–	-5

Introduce either x_1 or s_2. Note the first row is the negative of the last: phase I will complete if we push out u_1 adding x_1.

The next step is to introduce x_1 for u_1. This gives tableau T_3.

Note can check that $x_1 = \frac{5}{4}$ $s_1 = \frac{1}{2}$ $x_3 = \frac{19}{2}$ satisfy (*) with $z = 31$.

T_3	z'	z	x_1	x_2	x_3	s_1	s_2	u_1	u_2	
x_1	0	0	1	$-\frac{7}{4}$	0	0	$\frac{1}{4}_{[5]}$	–	–	$\frac{5}{4}$
s_1	0	0	0	$\frac{15}{2}_{[\frac{1}{15}]}$	0	1	$\frac{1}{2}_{[1]}$	–	–	$\frac{1}{2}$
x_3	0	0	0	$-\frac{5}{2}$	1	0	$-\frac{1}{2}$	–	–	$\frac{19}{2}$
II	0	1	0	-12	0	0	-1	–	–	31
I	1	0	0	0	0	0	0	–	–	0

Now we work with the reduced costs for z. We have $z = 31 + 12x_2 + s_2$.

The first phase is complete, we have a basic feasible solution to the new problem, and so we enter phase II and drop the artificial variables. The next step is to introduce s_2 for s_1.

T_4	z	x_1	x_2	x_3	s_1	s_2	
x_1	0	1	$-\frac{11}{2}$	0	$-\frac{1}{2}$	0	1
s_2	0	0	15	0	2	1	1
x_3	0	0	5	1	1	0	10
	1	0	3	0	2	0	32

Every entry in the final row is positive, so this is an optimal solution, and corresponds to the solution $x_1 = 1, x_2 = 0, x_3 = 10$ (and $s_1 = 0, s_2 = 1$), and $z = 32$.