22/10/15. PROBLEMS CLASS : 3W2.1. (A)

2(a). $\underline{A}\,\underline{x} = \underline{b}$ EQUIVALENTLY WRITTEN AS $\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b}$

SO THAT $\underline{x}_B = \underline{B}^{-1}\underline{b} - \underline{B}^{-1}\underline{N}\,\underline{x}_N$

THE OBJECTIVE FUNCTION $\underline{c}^T\underline{x} = \underline{c}_B^T\underline{x}_B + \underline{c}_N^T\underline{x}_N$

$$= \underline{c}_B^T(\underline{B}^{-1}\underline{b} - \underline{B}^{-1}\underline{N}\,\underline{x}_N) + \underline{c}_N^T\underline{x}_N$$

$$= \underline{c}_B^T\underline{B}^{-1}\underline{b} + (\underline{c}_N^T - \underline{c}_B^T\underline{B}^{-1}\underline{N})\underline{x}_N$$

(b)(iv). $\underline{B} = (\underline{a}_2, \underline{a}_4, \underline{a}_5) = \begin{pmatrix} -5 & 0 & 0 \\ -3 & 1 & 0 \\ -8 & 0 & 1 \end{pmatrix}$

FROM QUESTION 4 OF SOLUTION SHEET TWO:

$\underline{B}^{-1} = \begin{pmatrix} -\frac{1}{5} & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{8}{5} & 0 & 1 \end{pmatrix}$ $\underline{B}^{-1}\underline{b} = \begin{pmatrix} 2 \\ 3 \\ 12 \end{pmatrix}$

CORRESPONDING BFS: $(x_1, x_2, x_3, x_4, x_5)^T = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 3 \\ 12 \end{pmatrix}$

WE HAVE $\underline{c}_B^T = (-80, 0, 0)$, $\underline{c}_N^T = (-40, 0)$

$$\underline{B}^{-1}\underline{N} = \begin{pmatrix} -\frac{1}{5} & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{8}{5} & 0 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{cc} \underline{a}_1 & \underline{a}_3 \end{array} \\ \begin{pmatrix} -10 & 1 \\ -2 & 0 \\ -2 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} 2 & -1/5 \\ 4 & -3/5 \\ 14 & -8/5 \end{pmatrix}$$

FOR $\underline{\underline{\text{ANY}}}$ $\underline{x} \in F_{(c)}$

$$\underline{c}^T\underline{x} = (-80 \;\; 0 \;\; 0)\begin{pmatrix} 2 \\ 3 \\ 12 \end{pmatrix} + \left[(-40 \;\; 0) - (-80 \;\; 0 \;\; 0)\begin{pmatrix} 2 & -1/5 \\ 4 & -3/5 \\ 14 & -8/5 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

(B)

$$= -160 + [(-40 \quad 0) - (-160 \quad 16)] \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

$$= -160 + 120 x_1 - 16 x_3 .$$

BFS HAS $x_1 = x_3 = 0$ AND $\underline{c}^T \underline{x}^* = -160$. ANY SOLUTION WITH $x_1 > 0$ HAS A LARGER VALUE OF THE OBJECTIVE FUNCTION.

4. IDEA BEHIND PROOFS THIS WEEK.

$\underline{x} \in F_{(c)}$ SO $\underline{A}\,\underline{x} = \underline{b}$. $\quad x_j = \begin{cases} > 0 & j = 1, \dots, k \\ 0 & j = k+1, \dots, m \end{cases}$

IF $\underline{x}$ IS NOT BASIC THEN $(\underline{a}_1, \dots, \underline{a}_k)$ LINEARLY DEPENDENT.

i.e. $\exists\, \alpha_j$ NOT ALL EQUAL TO ZERO SUCH THAT:

$$\sum_{j=1}^{k} \alpha_j \underline{a}_j = \underline{0}_m$$

SET $\underline{\alpha} = (\alpha_1, \dots, \alpha_k, 0, \dots, 0)$ THEN $\underline{A}\,\underline{\alpha} = \underline{0}$.

FOR SUITABLE CHOICE OF μ, $\underline{A}(\underline{x} \pm \mu \underline{\alpha}) = \underline{b}$ WITH $\underline{x} \pm \mu \underline{\alpha} \geq \underline{0}_n$. i.e. $\underline{x} \pm \mu \underline{\alpha} \in F_{(c)}$.

(a). $\underline{\alpha} = (0, -1/2, 5/6, 1, 2/3)$. NOTE:

$$-\frac{1}{2}\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} + \frac{5}{6}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

CONSIDER

$$\begin{array}{cccccc} j & 1 & 2 & 3 & 4 & 5 \\ \dfrac{x_j}{|\alpha_j|} = & \dfrac{1/2}{0}, & \dfrac{1}{|-1/2|}, & \dfrac{3/2}{5/6}, & \dfrac{1}{1}, & \dfrac{1/2}{2/3} \end{array}$$

$$= \quad \infty, \; 2, \; 1\tfrac{4}{5}, \; 1, \; 3/4$$

FOR $0 < \mu \leq 3/4$

$$\underline{x} \pm \mu \underline{\alpha} = \begin{pmatrix} 1/2 \\ 1 \\ 3/2 \\ 1 \\ 1/2 \end{pmatrix} \pm \mu \begin{pmatrix} 0 \\ -1/2 \\ 5/6 \\ 1 \\ 2/3 \end{pmatrix} \in F_{(c)}.$$

TAKE $\mu = 3/4$ $\quad \underline{x} - \mu \underline{\alpha} = \begin{pmatrix} 1/2 \\ 11/8 \\ 7/8 \\ 1/4 \\ 0 \end{pmatrix}, \quad \underline{x} + \mu \underline{\alpha} = \begin{pmatrix} 1/2 \\ 5/8 \\ 17/8 \\ 7/4 \\ 1 \end{pmatrix}$

FEASIBLE SOLUTION WITH ONE MORE ZERO $\rightarrow 0$

$$\frac{1}{2}(\underline{x} - \mu \underline{\alpha}) + \frac{1}{2}(\underline{x} + \mu \underline{\alpha}) \in F_{(c)}.$$

(b). USING $\underline{\alpha}$, GENERATE FEASIBLE SOLUTION $\underline{x}_1 = (1/2, \; 11/8, \; 7/8, \; 1/4, \; 0)$.

REPEATING USING $\underline{x}_1$ AND $\underline{\beta}$

$$\min \left\{ \frac{1/2}{|-4|} \; \frac{11/8}{|1|} \; \frac{7/8}{|-3|} \; \frac{1/4}{|-2|} \right\} = \min \left\{ \frac{1}{8}, \frac{11}{8}, \frac{7}{24}, \frac{1}{8} \right\}$$

FOR $\mu_1 \leq \frac{1}{8}$ $\quad \underline{x}_1 \pm \mu_1 \underline{\beta} \in F_{(c)}$ AS $\underline{A}\,\underline{\beta} = \underline{0}$.

TAKE $\mu_1 = 1/8$

$$\underline{x}_1 + \mu_1 \underline{\beta} = \begin{pmatrix} 1/2 \\ 11/8 \\ 7/8 \\ 1/4 \\ 0 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} -4 \\ 1 \\ -3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}$$

NOTE: $\underline{x}_2 = (0, 3/2, 1/2, 0, 0)^T \in F_{(C)}$ AND $\underline{a}_2, \underline{a}_3$ ARE LINEARLY INDEPENDENT.

THIS IS A BASIC FEASIBLE SOLUTION!
[ACTUALLY A DEGENERATE BFS AS $\text{Rank}(\underline{A}) = 3$.
$\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$, $\{\underline{a}_2, \underline{a}_3, \underline{a}_4\}$, $\{\underline{a}_2, \underline{a}_3, \underline{a}_5\}$ ALL LEAD TO THIS SOLUTION.

5. INTRODUCING SLACK VARIABLES DOESN'T INCREASE THE NUMBER OF EXTREME POINTS.

$\underline{x}_1 \in F_{(S)}$ WITH EQUIVALENT SOLUTION $\underline{x}_1' = \begin{pmatrix} \underline{x}_1 \\ \underline{s}_1 \end{pmatrix} \in F_{(C')}$

$\underline{x}_2 \in F_{(S)}$ WITH EQUIVALENT SOLUTION $\underline{x}_2' = \begin{pmatrix} \underline{x}_2 \\ \underline{s}_2 \end{pmatrix} \in F_{(C')}$

$F_{(S)}$ AND $F_{(C')}$ CONVEX: $\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2 \in F_{(S)}$
$\lambda \underline{x}_1' + (1-\lambda)\underline{x}_2' \in F_{(C')}$

THUS, LINE SEGMENTS IN $F_{(C')}$ CORRESPOND TO LINE SEGMENTS IN $F_{(S)}$ AND VICE VERSA.

i.e. NON-EXTREME POINTS CORRESPOND TO EACH OTHER AND EXTREME POINTS CORRESPOND TO EACH OTHER.