15/10/2015. PROBLEMS CLASS: 3WN2.1.

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4 minimise $z = 40x_1 + 80x_2$ subject to

$$10x_1 + 5x_2 \geq 10$$
$$2x_1 + 3x_2 \geq 3$$
$$2x_1 + 8x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

$\rightsquigarrow$ maximise $z' = -40x_1 - 80x_2$ subject to

$$-10x_1 - 5x_2 + s_1 \leq -10$$
$$-2x_1 - 3x_2 + s_2 \leq -3$$
$$-2x_1 - 8x_2 + s_3 \leq -4$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

PROBLEM IN CANONICAL FORM

$\underline{A}\,\underline{x} = \underline{b}$ WHERE

$$\underline{A} = \begin{array}{c} \begin{array}{ccccc} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 & \underline{a}_4 & \underline{a}_5 \end{array} \\ \begin{pmatrix} -10 & -5 & 1 & 0 & 0 \\ -2 & -3 & 0 & 1 & 0 \\ -2 & -8 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \underline{b} = \begin{pmatrix} -10 \\ -3 \\ -4 \end{pmatrix}$$

NOT THE PRESENCE OF $\underline{I}_3$. CAN READ OFF AN IMMEDIATE SOLUTION TO $\underline{A}\,\underline{x} = \underline{b}$

$[\ \underline{x} = (x_1, x_2, s_1, s_2, s_3)^T\ ]$

$s_1 = -10 \quad s_2 = -3 \quad s_3 = -4 \quad x_1 = 0, \quad x_2 = 0$

THIS IS A BASIC SOLUTION: BASIS $\{s_1, s_2, s_3\}$

NON-BASIS $\{x_1, x_2\}$

- COLUMNS OF $\underline{A}$ CORRESPONDING TO BASIS VARIABLES $(\underline{a}_3, \underline{a}_4, \underline{a}_5)$ LINEARLY INDEPENDENT
- NON-BASIC VARIABLES ARE EQUAL TO ZERO
- $\underline{A}\,\underline{x} = \underline{b}$ BUT $\underline{x} \ngeq \underline{0}_5$ SO NOT FEASIBLE.

RANK$(\underline{A}) = 3 \quad n = 5, \ m = 3$ FOR THE PROBLEM IN CANONICAL FORM. THERE ARE $\binom{5}{3} = 10$ WAYS OF CHOOSING THREE COLUMNS.

IF THE CHOSEN THREE COLUMNS ARE LINEARLY INDEPENDENT, CAN FIND A BASIC SOLUTION.

CHOOSE $\{x_1, x_2, s_1\}$ AS BASIS, $\{s_2, s_3\}$ NON-BASIC VARIABLES.

IDEA: PARTITION $\underline{A} = (\underline{B} \mid \underline{N})$ WHERE $\underline{B}$ IS INVERTIBLE $\underline{x} = \begin{pmatrix} \underline{x}_B \\ \underline{x}_N \end{pmatrix}$

$\underline{A}\,\underline{x} = \underline{b}$ CORRESPONDS TO $\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b}$

BASIS $\{s_1, s_2, s_3\}$: TAKE $\underline{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\underline{N} = \begin{pmatrix} -10 & -5 \\ -2 & -3 \\ -2 & -8 \end{pmatrix}$

$\underline{x}_B = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$ $\underline{x}_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underline{B}^{-1}\underline{b} = \underline{b} = \begin{pmatrix} -10 \\ -3 \\ -4 \end{pmatrix}$

FOR THE BASIS $\{x_1, x_2, s_1\}$ $\underline{B} = \begin{pmatrix} -10 & -5 & 1 \\ -2 & -3 & 0 \\ -2 & -8 & 0 \end{pmatrix}$ $\underline{N} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\underline{B}^{-1} = \begin{pmatrix} 0 & -\frac{4}{5} & \frac{3}{10} \\ 0 & \frac{1}{5} & -\frac{1}{5} \\ 1 & -7 & 2 \end{pmatrix}$ $\underline{B}^{-1}\underline{N} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{10} \\ \frac{1}{5} & -\frac{1}{5} \\ -7 & 2 \end{pmatrix}$ $\underline{B}^{-1}\underline{b} = \begin{pmatrix} 6/5 \\ 1/5 \\ 3 \end{pmatrix}$

EQUIVALENT SYSTEM TO $\underline{A}\,\underline{x} = \underline{b}$ IS $\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b}$
WHICH IS:

$$\begin{aligned}
x_1 - \tfrac{4}{5}s_2 + \tfrac{3}{10}s_3 &= 6/5 \Rightarrow x_1 = 6/5 + 4/5\,s_2 - 3/10\,s_3 \\
x_2 + \tfrac{1}{5}s_2 - \tfrac{1}{5}s_3 &= 1/5 \Rightarrow x_2 = 1/5 - 1/5\,s_2 + 1/5\,s_3 \\
s_1 - 7s_2 + 2s_3 &= 3 \Rightarrow s_1 = 3 + 7s_2 - 2s_3
\end{aligned}$$

$\underline{x}_B = \underline{B}^{-1}\underline{b} - \underline{B}^{-1}\underline{N}\,\underline{x}_N$

AN IMMEDIATE SOLUTION IS

$x_1 = 6/5$ $x_2 = 1/5$ $s_1 = 3$, $s_2 = 0$, $s_3 = 0$

THIS IS THE BASIC SOLUTION CORRESPONDING TO THE BASIS $\{x_1, x_2, s_1\}$

(c)

AS $\underline{x} \geq \underline{0}_5$ THIS IS ALSO A BASIC FEASIBLE SOLUTION.

NOW, WE KNOW $x_1 = 6/5$ $x_2 = 1/5$ IS THE POINT AT WHICH THE STANDARD PROBLEM IS OPTIMISED.
SO, $x_1 = 6/5$, $x_2 = 1/5$, $s_1 = 3$, $s_2 = 0$, $s_3 = 0$ IS OPTIMAL FOR THE CANONICAL PROBLEM. HOW CAN WE SEE THIS?

$$z' = -40x_1 - 80x_2$$

EXPRESS OBJECTIVE FUNCTION IN TERMS OF THE NON-BASIC VARIABLES.

$$z' = -40\left(6/5 + 4/5\, s_2 - 3/10\, s_3\right) - 80\left(1/5 - 1/5\, s_2 + 1/5\, s_3\right)$$
$$= -64 - 16 s_2 - 4 s_3$$

CURRENT BFS HAS $s_2 = 0$, $s_3 = 0$ AND SO OBJECTIVE FUNCTION HAS THE VALUE -64.

ANY FEASIBLE SOLUTION WHICH DOES NOT HAVE $s_2 = 0$ AND $s_3 = 0$ HAS A SMALLER VALUE OF THE OBJECTIVE FUNCTION.

THUS, THE CURRENT BASIC FEASIBLE SOLUTION $(6/5, 1/5, 3, 0, 0)$ IS OPTIMAL WITH $z' = -64$.

$\binom{5}{3} = 10$ POSSIBLE BASIC SOLUTIONS.

FOUR OF THEM ARE FEASIBLE: $\{x_1, s_2, s_3\}$, $\{x_1, x_2, s_3\}$, $\{x_1, x_2, s_1\}$, $\{x_2, s_1, s_2\}$

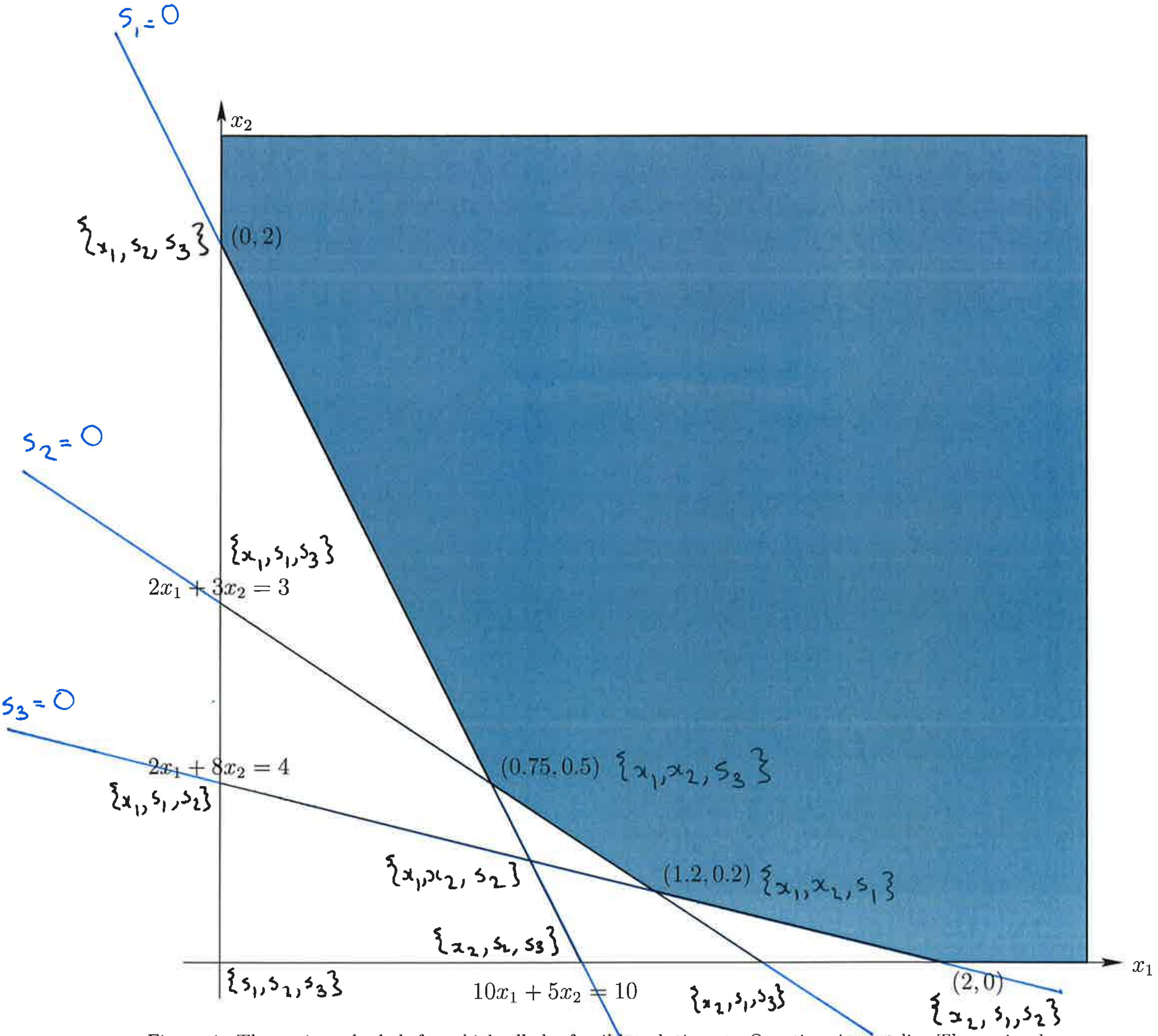

Figure 1: The region, shaded, for which all the feasible solutions to Question 4 must lie. The region has four extreme points: (0, 2), (0.75, 0.5), (1.2, 0.2) and (2, 0).

I DIDN'T COVER THIS IN THE PROBLEMS CLASS BUT HERE'S ANOTHER BASIS:

CHOOSE $\{x_1, x_2, s_3\}$ AS BASIS, $\{s_1, s_2\}$ NON-BASIC.

$$\underline{B} = \begin{pmatrix} -10 & -5 & 0 \\ -2 & -3 & 0 \\ -2 & -8 & 1 \end{pmatrix} \quad \underline{B}^{-1} = \begin{pmatrix} -\frac{3}{20} & \frac{1}{4} & 0 \\ \frac{1}{10} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{7}{2} & 1 \end{pmatrix} \quad \underline{B}^{-1}\underline{b} = \begin{pmatrix} 3/4 \\ 1/2 \\ 3/2 \end{pmatrix}$$

$$\underline{N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \underline{B}^{-1}\underline{N} = \begin{pmatrix} -\frac{3}{20} & \frac{1}{4} \\ \frac{1}{10} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{7}{2} \end{pmatrix}$$

EQUIVALENT SYSTEM TO $\underline{A}\,\underline{x} = \underline{b}$ IS:

$$(B) \quad \begin{aligned} x_1 - \tfrac{3}{20}s_1 + \tfrac{1}{4}s_2 &= 3/4 \qquad (*) \\ x_2 + \tfrac{1}{10}s_1 - \tfrac{1}{2}s_2 &= 1/2 \qquad (**) \\ s_3 + \tfrac{1}{2}s_1 - \tfrac{7}{2}s_2 &= 3/2 \end{aligned}$$

SO THAT $x_1 = 3/4$, $x_2 = 1/2$, $s_3 = 3/2$, $s_1 = 0$, $s_2 = 0$ IS AN IMMEDIATE SOLUTION. IT IS THE BFS CORRESPONDING TO THE BASIS $\{x_1, x_2, s_3\}$ AND IS A BASIC FEASIBLE SOLUTION.

ANY FEASIBLE SOLUTION SATISFIES (B). SUBSTITUTE (*) AND (**) INTO z' TO GIVE:

$$\begin{aligned} z' = -40x_1 - 80x_1 &= -40\left(3/4 + 3/20\,s_1 - 1/4\,s_2\right) \\ &\quad -80\left(1/2 - 1/10\,s_1 + 1/2\,s_2\right) \\ &= -70 + 2s_1 - 30s_2. \end{aligned}$$

THIS SHOWS THAT $(3/4, 1/2, 0, 0, 3/2)$ IS NOT OPTIMAL AS A SOLUTION WITH $s_1 > 0$ WOULD INCREASE z'.

[NOTE: AT $(6/5\ \ 1/5\ \ 3\ \ 0\ \ 0)$ WE HAVE $z = -70 + 2(3) - 30(0)$
$= -64$ AS EXPECTED].