8/10/2015. PROBLEMS CLASS: 3WN2.1. (A)

5. (S) maximise $z = \underline{c}^T \underline{x}$ subject to $\underline{A}\,\underline{x} \leq \underline{b}$, $\underline{x} \geq \underline{0}_n$

(C) maximise $z = \underline{c}'^T \underline{x}'$ subject to $\underline{A}'\,\underline{x}' = \underline{b}$, $\underline{x}' \geq \underline{0}_{n+m}$

WHERE $\underline{A}' = (\underline{A} \,|\, \underline{I}_m)$ $\underline{x}' = \begin{pmatrix} \underline{x} \\ \underline{s} \end{pmatrix}$ $\underline{c}' = \begin{pmatrix} \underline{c} \\ \underline{0}_m \end{pmatrix}$

i.e. ADD IN SLACK VARIABLES.

LET $F_{(S)} = \{ \underline{x} \in \mathbb{R}^n : \underline{A}\,\underline{x} \leq \underline{b},\ \underline{x} \geq \underline{0}_n \}$ FEASIBLE SET FOR (S)
$F_{(C)} = \{ \underline{x}' \in \mathbb{R}^{n+m} : \underline{A}'\,\underline{x}' = \underline{b},\ \underline{x}' \geq \underline{0}_{n+m} \}$ FEASIBLE SET FOR (C)

STEP ONE: FEASIBLE SOLUTION TO (S) $\Rightarrow$ FEASIBLE SOLUTION TO (C) WITH SAME VALUE OF OBJECTIVE FUNCTION.
OPTIMAL SOLUTION OF (S) LOWER BOUND TO OPTIMAL SOLUTION OF (C)

$\underline{x} \in F_{(S)}$. LET $\underline{s} = \underline{b} - \underline{A}\,\underline{x} \geq \underline{0}_m$ SO $\underline{A}\,\underline{x} + \underline{s} = \underline{b}$. THUS, $\underline{x}' = \begin{pmatrix} \underline{x} \\ \underline{s} \end{pmatrix} \in F_{(C)}$ WITH $\underline{c}'^T \underline{x}' = \underline{c}^T \underline{x} + \underline{0}_m^T \underline{s} = \underline{c}^T \underline{x}$.

HENCE $$\sup_{\underline{x} \in F_{(S)}} \underline{c}^T \underline{x} \leq \sup_{\underline{x}' \in F_{(C)}} \underline{c}'^T \underline{x}'. \qquad (*)$$

[AS MIGHT BE $\underline{x}' \in F_{(C)}$ NOT CONSTRUCTED IN THIS WAY].

STEP TWO: FEASIBLE SOLUTION TO (C) $\Rightarrow$ FEASIBLE SOLUTION TO (C) WITH SAME VALUE OF OBJECTIVE FUNCTION
OPTIMAL SOLUTION OF (C) LOWER BOUND TO OPTIMAL SOLUTION OF (S)

$\underline{x}' \in F_{(C)}$ THEN $\underline{x}' \geq \underline{0}_{n+m}$. IN PARTICULAR, IF $\underline{x}' = \begin{pmatrix} \underline{x} \\ \underline{s} \end{pmatrix}$ THEN

$\underline{x} \geq \underline{0}_n$ AND $\underline{s} \geq \underline{0}_m$ AND $\underline{A}'\,\underline{x}' = \underline{A}\,\underline{x} + \underline{s} = \underline{b}$

THUS, $\underline{A}\,\underline{x} \le \underline{b}$ AND SO $\underline{x} \in F_{(S)}$. ADDITIONALLY

$$\underline{c}'^{T} \underline{x}' = \underline{c}^{T}\underline{x} + \underline{0}_m^{T} \underline{s} = \underline{c}^{T}\underline{x} .$$

HENCE,

$$\sup_{\underline{x}' \in F_{(L)}} \underline{c}'^{T} \underline{x}' \le \sup_{\underline{x} \in F_{(S)}} \underline{c}^{T} \underline{x} \qquad (**)$$

COMBINING (*) AND (**) GIVES THE RESULT: IF $\underline{x} \in F_{(S)}$ IS OPTIMAL FOR (C) THEN $\begin{pmatrix} \underline{x} \\ \underline{s} = \underline{b} - \underline{A}\,\underline{x} \end{pmatrix}$ OPTIMAL FOR (S) AND VICE VERSA.

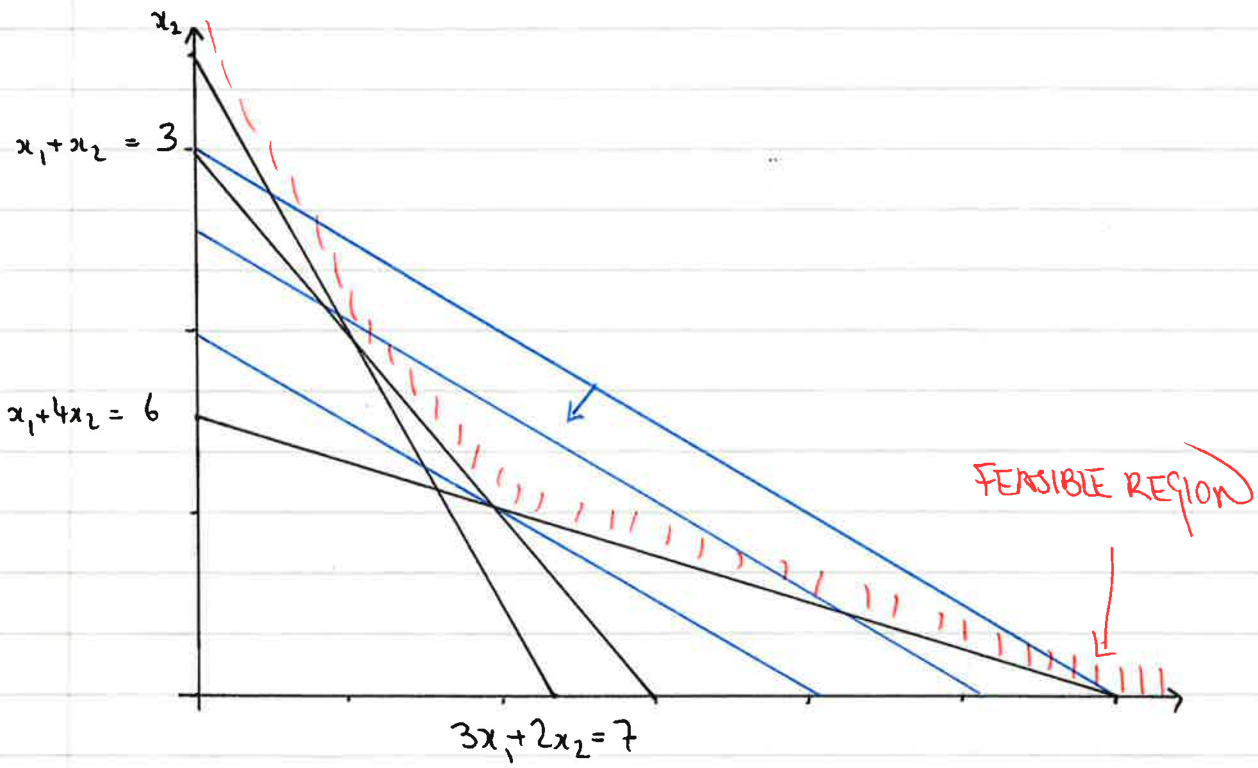
4.

a. PLOT THE CONSTRAINTS.

$x_1 + x_2 = 3$ INTERCEPTS ARE $(0, 3)$ AND $(3, 0)$

$x_1 + 4x_2 = 6$ INTERCEPTS ARE $(0, 3/2)$ AND $(6, 0)$

$3x_1 + 2x_2 = 7$ INTERCEPTS ARE $(0, 7/2)$ AND $(7/3, 0)$



c.

NOTE: (0, 0) IS NOT FEASIBLE FOR ANY OF THE CONSTRAINTS.

FEASIBLE REGION IS UNBOUNDED

OBJECTIVE FUNCTION $z = x_1 + 2x_2$. CHOOSE A POINT e.g. (0, 3) THEN $z = 6$. $x_1 + 2x_2 = 6$ CROSSES AT (6, 0).

MINIMISING $x_1 + 2x_2$ CORRESPONDS TO MOVING IN A S-W DIRECTION (DECREASING x_1, x_2)

MINIMUM OCCURS WHEN $x_1 + x_2 = 3$ AND $x_1 + 4x_2 = 6$

$\Rightarrow \quad x_1 = 2 \quad x_2 = 1$

MINIMUM IS (2, 1) WHEN $z = 4$.

PROBLEM IN STANDARD FORM AND CANONICAL FORM

(S) maximum $z = -x_1 - 2x_2$

subject to

$-x_1 - x_2 \leq -3$ TIGHT

$-x_1 - 4x_2 \leq -6$ TIGHT

$-3x_1 - 2x_2 \leq -7$

$x_1, x_2 \geq 0$

minimum $z = -x_1 - 2x_2$

subject to $-x_1 - x_2 + s_1 = -3$

$-x_1 - 4x_2 + s_2 = -6$

$-3x_1 - 2x_2 + s_3 = -7$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

OPTIMAL SOLUTION $x^* = (x_1^*, x_2^*) = (2, 1)$ WITH $z = 4$

OPTIMAL SOLUTION

$x'^* = (x_1^*, x_2^*, s_1^*, s_2^*, s_3^*)$

$= (2, 1, 0, 0, 1)$

THIS HAS 3 NON-ZERO ELEMENTS AND 2 ZERO ELEMENTS.

THE PROBLEM HAS 3 CONSTRAINTS SO AS WE'LL SHOW THIS IS NOT A SURPRISE.