1/10/2015. PROBLEMS CLASS: BWN2.1. (A)

1.

$$\begin{aligned} \text{minimise} \quad & z = 40x_1 + 80x_2 \\ \text{subject to} \quad & 10x_1 + 5x_2 \geq 10 \\ & 2x_1 + 3x_2 \geq 3 \\ & 2x_1 + 8x_2 \geq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a). SOLVING USING THE GRAPHICAL METHOD.

1. CREATE THE FEASIBLE REGION → FIGURE 1
2. MOVE THE OBJECTIVE FUNCTION $z = 40x_1 + 80x_2$ THROUGH THE FEASIBLE REGION IN A DIRECTION THAT DECREASES z (AS A MINIMISATION PROBLEM!)
   [DECREASING x_1, x_2 DECREASES $z \Rightarrow$ SOUTH-WEST DIRECTION].
   PICK A FEASIBLE VALUE OF z e.g. $(0, 2)$.
   → FIGURE 2.

(b) WRITING AS A STANDARD MAXIMISATION PROBLEM:

$$\begin{aligned} \text{maximise} \quad & z' = -40x_1 - 80x_2 \\ \text{subject to} \quad & -10x_1 - 5x_2 \leq -10 \\ & -2x_1 - 3x_2 \leq -3 \\ & -2x_1 - 8x_2 \leq -4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

$$\underline{c} = \begin{pmatrix} -40 \\ -80 \end{pmatrix}, \quad \underline{A} = \begin{pmatrix} -10 & -5 \\ -2 & -3 \\ -2 & -8 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} -10 \\ -3 \\ -4 \end{pmatrix}$$

(2). ① CREATE THE FEASIBLE REGION ⇒ FIGURE 3.

② MOVE THE OBJECTIVE FUNCTION $z = 18x_1 + 6x_2$ THROUGH THE FEASIBLE REGION IN A N-E DIRECTION (AS MAXIMISING!) [INCREASING x_1, x_2 INCREASES z].

3. IF WE WRITE THE CONSTRAINTS IN MAXIMISATION FORM WE HAVE:

$$\left.\begin{aligned} x_1 + x_2 &\leq 4 \\ -x_1 - x_3 &\leq -7 \end{aligned}\right\} \text{ADDING THESE: } \quad x_2 - x_3 \leq -3 \quad \Rightarrow \quad x_3 - x_2 \geq 3.$$

$$x_3 - x_2 \leq 1$$

4.a. $4x_1 + 2x_2 - x_3 = 5$ EQUIVALENT TO: $4x_1 + 2x_2 - x_3 \leq 5$

$4x_1 + 2x_2 - x_3 \geq 5$

EQUIVALENT TO: $4x_1 + 2x_2 - x_3 \leq 5$

$-4x_1 - 2x_2 + x_3 \leq -5$

$x_1 + 3x_2 \geq 2$ EQUIVALENT TO $-x_1 - 3x_2 \leq -2$

SO, PROBLEM EQUIVALENT TO: maximise $z = 3x_1 + 4x_2 + x_3$

subject to

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &\leq 5 \\ -4x_1 - 2x_2 + x_3 &\leq -5 \\ -x_1 - 3x_2 &\leq -2 \\ x_2 - x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

i.e. $\underline{c} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ $\quad \underline{A} = \begin{pmatrix} 4 & 2 & -1 \\ -4 & -2 & 1 \\ -1 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 5 \\ -5 \\ -2 \\ 3 \end{pmatrix}$

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b. SUPPOSE $x_1 \geq 1$ THEN $y = x_1 - 1 \geq 0$. SUBSTITUTE $x_1 = y+1$ EVERYWHERE:

$$\begin{aligned}
\text{maximise} \quad & z = 3(y+1) + 4x_2 + x_3 \\
\text{subject to} \quad & 4(y+1) + 2x_2 - x_3 \leq 5 \\
& -4(y+1) - 2x_2 + x_3 \leq -5 \\
& -(y+1) - 3x_2 \leq -2 \\
& x_2 - x_3 \leq 3 \\
& y, x_2, x_3 \geq 0
\end{aligned}$$

AN EQUIVALENT PROBLEM IS THEN:

$$\begin{aligned}
\text{maximise} \quad & z' = 3y + 4x_2 + x_3 \\
\text{subject to} \quad & 4y + 2x_2 - x_3 \leq 1 \\
& -4y - 2x_2 + x_3 \leq -1 \\
& -y - 3x_2 \leq -1 \\
& x_2 - x_3 \leq 3 \\
& y, x_2, x_3 \geq 0
\end{aligned}$$

c. SUPPOSE INSTEAD THAT x_3 UNCONSTRAINED. WRITE $x_3 = w_3 - y_3$ WHERE $w_3 \geq 0$, $y_3 \geq 0$. SUBSTITUTING EVERYWHERE GIVES:

$$\begin{aligned}
\text{maximise} \quad & z = 3x_1 + 4x_2 + w_3 - y_3 \\
\text{subject to} \quad & 4x_1 + 2x_2 - w_3 + y_3 \leq 5 \\
& -4x_1 - 2x_2 + w_3 - y_3 \leq -5 \\
& -x_1 - 3x_2 \leq -2 \\
& x_2 - w_3 + y_3 \leq 3 \\
& x_1, x_2, w_3, y_3 \geq 0.
\end{aligned}$$

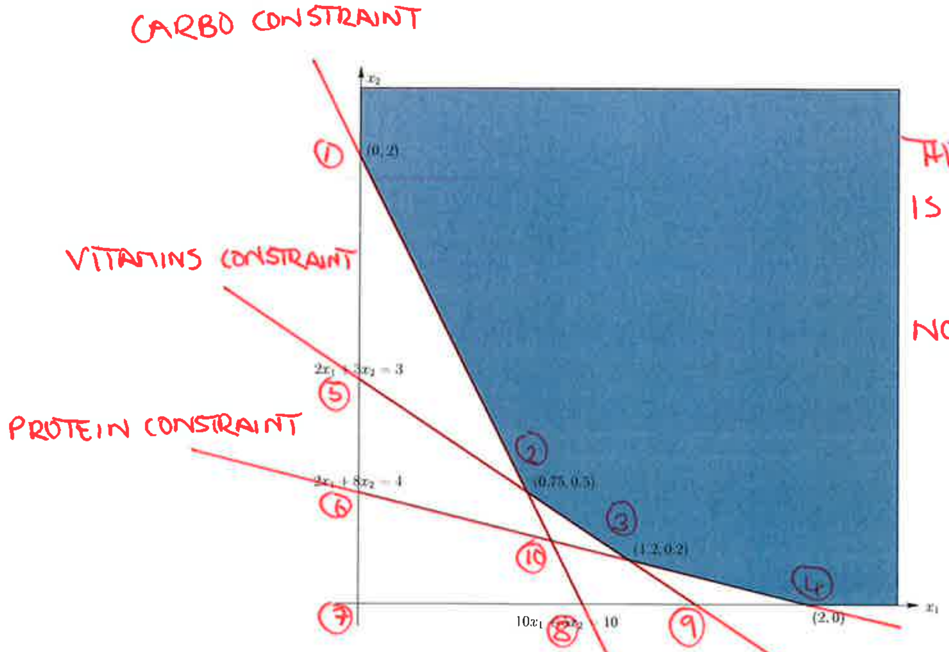
Example 1 *(Cost minimisation problem) We wish to feed cattle, meeting nutritional requirements at minimum cost. A farmer has two feeds "Mootastic" and "Udderly Wonderful" at her disposal. There are certain nutritional requirements which stipulate that cattle should receive minimum quantities of carbohydrate, vitamins and protein. The table below gives the number of units per kilo of the nutrients required along with the cost of the two feeds.*

Food	*Carbo.*	*Vit.*	*Pro.*	*Cost (pence/kilo)*
Mootastic	*10*	*2*	*2*	*40*
Udderly Wonderful	*5*	*3*	*8*	*80*
Daily Requirement	*10*	*3*	*4*	

Let x_1 denote the number of kilos of Mootastic and x_2 the number of kilos of Udderly Wonderful used per day. Our objective is to:

$$\begin{array}{rll} \textit{minimise} & z = 40x_1 + 80x_2 & \\ \textit{subject to} & 10x_1 + 5x_2 \geq 10 & \text{CARBO} \\ & 2x_1 + 3x_2 \geq 3 & \text{VITAMINS} \\ & 2x_1 + 8x_2 \geq 4 & \text{PROTEIN} \\ & x_1, x_2 \geq 0. & \end{array}$$

This is a linear programming problem with two variables and three constraints. As this is a problem in $\mathbb{R}^2$ we can solve the problem graphically. In Figure 1 we illustrate the feasible region, that is the set of points with coordinates (x_1, x_2) that satisfy all of the constraints.



THE FEASIBLE REGION IS UNBOUNDED.

NOTE: (0,0) DOES NOT SATISFY ANY OF THE CONSTRAINTS.

Figure 1: The region, shaded, for which all the feasible solutions to Example 1 must lie. The region has four extreme points: $(0, 2)$, $(0.75, 0.5)$, $(1.2, 0.2)$ and $(2, 0)$. The lines $x_1 = 0$ (for $x_2 \geq 2$) and $x_2 = 0$ (for $x_1 \geq 2$) are **extreme rays**. This feasible set is unbounded.

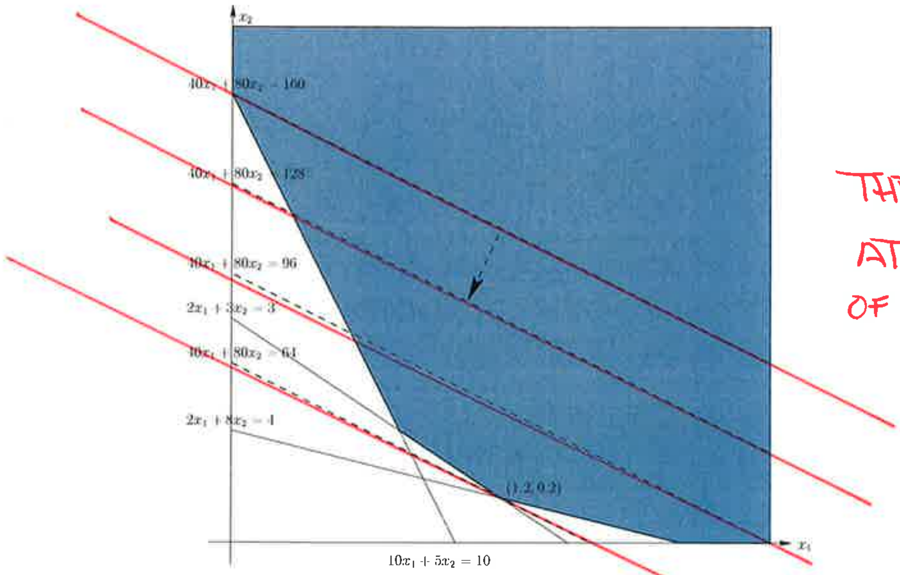
The objective is to minimise the linear function $z = 40x_1 + 80x_2$. Figure 2 illustrates how we do this.

INTERSECTION OF CONSTRAINTS OUTSIDE OF THE FEASIBLE SET:

POINTS 5 – 10 THESE WILL LATER CORRESPOND TO "BASIC SOLUTIONS" WHICH ARE NOT FEASIBLE.

ONLY FOUR OF THE TEN INTERSECTIONS OF THE CONSTRAINTS CORRESPOND TO FEASIBLE SOLUTIONS: POINTS 1 – 4.

THESE WILL LATER CORRESPOND TO "BASIC FEASIBLE SOLUTIONS"



THE MINIMUM OCCURS AT AN EXTREME POINT OF THE FEASIBLE SET.

Figure 2: Translating the objective function of Example 1 through the feasible region. The minimum of 64 occurs at the extreme point $(1.2, 0.2)$.

If we fix a value of z then $40x_1 + 80x_2 \leq z$ corresponds to points below the line. Decreasing the value of z corresponds to translating the objective function through the feasible region in a southwest direction. For example, Figure 2 shows the objective function for decreasing values of z. Note that when $z = 64$ we obtain the minimum at the extreme point $(1.2, 0.2)$ where $z = 64$. This extreme point corresponds to $2x_1 + 3x_2 = 3$ and $2x_1 + 8x_2 = 4$: the constraints $2x_1 + 3x_2 \geq 3$ and $2x_1 + 8x_2 \geq 4$ are ***active*** *or* ***binding*** *or* ***tight*** *at this point. Note that whilst the feasible region is unbounded, there was a unique solution to the problem.*[1]

OPTIMAL SOLUTION $z = 64$ AT $(1.2, 0.2)$.

$2x_1 + 3x_2 = 3$
$2x_1 + 8x_2 = 4$ ACTIVE/BINDING/TIGHT AT $x_1 = 1.2$, $x_2 = 0.2$.

$10x_1 + 5x_2 > 10$ SLACK.

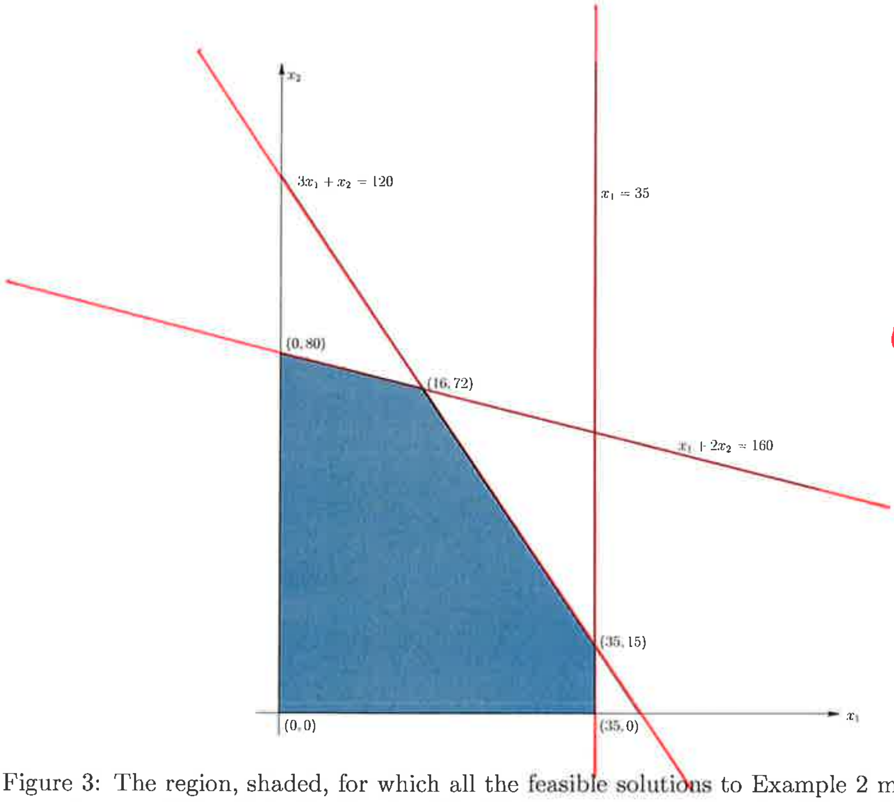
$10(1.2) + 5(0.2) = 13 \geq 10$ SO FEASIBLE!

[1]Note also that had we sought to maximise $z = 40x_1 + 80x_2$ over the feasible region we would have found that there was no optimal solution.

Example 2 *Consider the linear programming problem:*

$$\begin{aligned} \textit{maximise} \quad & z = 18x_1 + 6x_2 \\ \textit{subject to} \quad & 3x_1 + x_2 \leq 120 \\ & x_1 + 2x_2 \leq 160 \\ & x_1 \leq 35 \\ & x_1, x_2 \geq 0. \end{aligned}$$

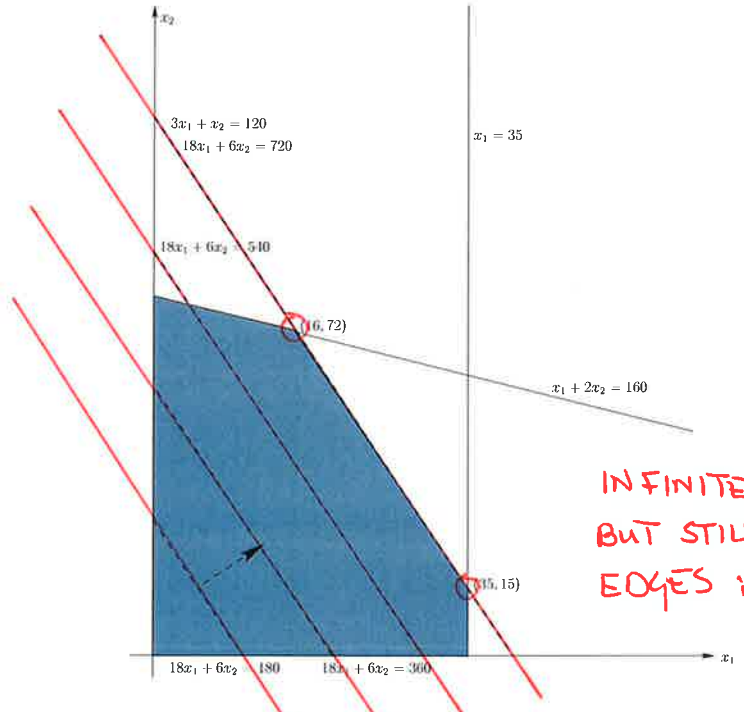
Once again this is a problem with two variables and three constraints. Consequently, we can solve this $\mathbb{R}^2$ *problem graphically. In Figure 3 we illustrate the feasible region, that is the set of points with coordinates* (x_1, x_2) *that satisfy all of the constraints.*



NOTE: (0,0) IS FEASIBLE.

Figure 3: The region, shaded, for which all the feasible solutions to Example 2 must lie. The region has five extreme points: $(0,0)$, $(0,80)$, $(16,72)$, $(35,15)$ and $(35,0)$. This feasible set is bounded.

The objective is to maximise the linear function $z = 18x_1 + 6x_2$. *Figure 4 illustrates how we do this.*



INFINITE NUMBER OF SOLUTIONS BUT STILL INCLUDES TWO LINE EDGES i.e. EXTREME POINTS.

Figure 4: Translating the objective function of Example 2 through the feasible region. The maximum of 720 occurs is not unique, but does occur at the extreme points $(16, 72)$ and $(35, 15)$.

If we fix a value of z then $18x_1 + 6x_2 \geq z$ corresponds to points above the line. Increasing the value of z corresponds to translating the objective function through the feasible region in a northeast direction. For example, Figure 4 shows the objective function for increasing values of z. Note that when $z = 720$ we obtain the maximum but there are an infinite number of points where this is achieved. These do include two extreme points: the point $(16, 72)$, where the constraints $x_1 + 2x_2 \leq 160$ and $3x_1 + x_2 \leq 120$ are active, and the point $(35, 15)$, where the constraints $x_1 \leq 35$ and $3x_1 + x_2 \leq 120$ are active.

OPTIMAL SOLUTION EXISTS AND CAN BE FOUND AT AN EXTREME POINT THOUGH IT IS NOT UNIQUE IN THIS CASE.