26/10/2015. LECTURE 9: 3WN2.1. (A).

LAST TIME: $x_j = \begin{cases} > 0 & j=1,\dots,k \\ 0 & j=k+1,\dots,n \end{cases}$

SUPPOSE $\underline{a}_1,\dots,\underline{a}_k$ DEPENDENT THEN α_j NOT ALL EQUAL TO ZERO SUCH THAT $\sum_{j=1}^{k} \alpha_j \underline{a}_j = \underline{0}_m$

$\underline{\underline{\mu > 0}}$ SUCH THAT $\mu|\alpha_j| \le x_j$ FOR ALL $j=1,\dots,k$ THEN

$$\underline{x} - \mu\underline{\alpha},\ \underline{x} + \mu\underline{\alpha} \in F_{(1)} \text{ WHERE } \underline{\alpha} = (\alpha_1,\dots,\alpha_k,0,\dots,0)^T$$

AS $\underline{x}$ IS OPTIMAL,

$$\underline{c}^T\underline{x} \ge \underline{c}^T(\underline{x} - \mu\underline{\alpha}) = \underline{c}^T\underline{x} - \mu\underline{c}^T\underline{\alpha}$$

i.e. $\underline{\underline{\mu\underline{c}^T\underline{\alpha} \ge 0}}$ (*)

$$\underline{c}^T\underline{x} \ge \underline{c}^T(\underline{x} + \mu\underline{\alpha}) = \underline{c}^T\underline{x} + \mu\underline{c}^T\underline{\alpha}$$

i.e. $\underline{\underline{\mu\underline{c}^T\underline{\alpha} \le 0}}$ (**)

(*), (**) $\Rightarrow$ $\mu\underline{c}^T\underline{\alpha} = 0$. AS $\mu > 0$ THEN $\underline{\underline{\underline{c}^T\underline{\alpha} = 0}}$ AND $\underline{x} - \mu\underline{\alpha}$ AND $\underline{x} + \mu\underline{\alpha}$ ARE ALSO OPTIMAL.

TAKE $\mu = \left\{ \min_{j=1,\dots,k} \frac{x_j}{|\alpha_j|} : \alpha_j \ne 0 \right\}$ [EQUALLY $\mu = \infty$ IF $\alpha_j = 0$]

[SO, $\mu|\alpha_j| \le x_j$ FOR ALL $j=1,\dots,k$ WITH EQUALITY FOR AT LEAST ONE POINT].

LET j' BE SUCH THAT $\frac{x_{j'}}{|\alpha_{j'}|} = \mu$ THEN IF $\alpha_{j'} > 0$ $\underline{x} - \mu\underline{\alpha}$ IS AN OPTIMAL SOLUTION WITH AT MOST $k-1$ NON-ZERO VALUES.

IF $\alpha_j' < 0$ THEN $x + \mu\alpha$ IS AN OPTIMAL SOLUTION WITH AT MOST $k-1$ NON-ZERO VALUES.

THUS, THERE IS AN OPTIMAL SOLUTION WITH AT MOST $k-1$ NON-ZERO VALUES. REPEAT THIS ARGUMENT WITH THIS VALUE UNTIL COLUMNS CORRESPONDING TO NON-ZERO VARIABLES ARE LINEARLY INDEPENDENT.

USE THIS TRICK TO FIND A BFS FROM ANY FEASIBLE POINT

[EXAMPLE: QUESTION SHEET THREE 1(e) DO THIS IN ONE STEP.
QUESTION SHEET THREE 4(b) DO THIS IN TWO STEPS].

AT THIS POINT, WE GET AN OPTIMAL BASIC FEASIBLE SOLUTION.

CONCLUSION.

IF THERE EXISTS AN OPTIMAL SOLUTION, WE CAN FIND IT BY RESTRICTING ATTENTION TO BFS ≡ EXTREME POINTS

SO, PROBLEM WITH n VARIABLES, m CONSTRAINTS CAN BE SOLVED BY VISITING FINITELY MANY BFS. THERE ARE AT MOST $\binom{n}{m}$ OF THESE.

HOWEVER, $\binom{n}{m}$ IS LARGE! e.g. $n = 500$, $m = 300$: $\binom{500}{300} \simeq 10^{144}$

A MORE SYSTEMATIC APPROACH IS NEEDED. THIS IS THE SIMPLEX ALGORITHM.

4. THE SIMPLEX METHOD.

THE METHOD IS AN ITERATIVE PROCEDURE FOR MOVING FROM EXTREME POINT TO ADJACENT EXTREME POINT WITH AN IMPROVED VALUE OF THE OBJECTIVE FUNCTION. IT IS APPLIED TO PROBLEMS IN CANONICAL FORM.

③

4.1 PRELIMINARY THEORY.

WE SEEK $\underline{x} \in \mathbb{R}^n$ THAT WILL

$$\begin{aligned} &\text{maximise} \quad z = \underline{c}^T \underline{x} \\ &\text{subject to} \quad \underline{A}\,\underline{x} = \underline{b} \\ &\qquad\qquad\quad \underline{x} \geq \underline{0}_n \end{aligned}$$

WHERE $\underline{A} \in \mathbb{R}^{m\times n}$, $\text{Rank}(\underline{A}) = m$, $\underline{c} \in \mathbb{R}^n$, $\underline{x} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^m$, $F_{(c)}$ THE FEASIBLE SET

LET $\underline{B}$ BE $m\times m$ MATRIX WHOSE COLUMNS ARE A SUBSET OF THE COLUMNS OF $\underline{A}$ WHICH ARE LINEARLY INDEPENDENT. WLOG, ASSUME $\underline{B} = (\underline{a}_1, \dots, \underline{a}_m)$ WITH $\underline{N} = (\underline{a}_{m+1}, \dots, \underline{a}_n)$ SO THAT $\underline{A} = (\underline{B} \mid \underline{N})$. LET $\underline{x} = \begin{pmatrix} \underline{x}_B \\ \underline{x}_N \end{pmatrix}$ WITH $\underline{x}_B \in \mathbb{R}^m$, $\underline{x}_N \in \mathbb{R}^{n-m}$. THEN

$$\underline{A}\,\underline{x} = (\underline{B} \mid \underline{N}) \begin{pmatrix} \underline{x}_B \\ \underline{x}_N \end{pmatrix} = \underline{B}\,\underline{x}_B + \underline{N}\,\underline{x}_N$$

AS $\underline{B}$ IS INVERTIBLE, SOLUTIONS TO $\underline{A}\,\underline{x} = \underline{b}$ ARE OF THE FORM:

$$\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b} \qquad (*)$$

NOTE: $\underline{x}_N = \underline{0}_{n-m}$ GIVES $\underline{x}^* = \begin{pmatrix} \underline{B}^{-1}\underline{b} \\ \underline{0}_{n-m} \end{pmatrix}$, BASIC SOLUTION ASSOCIATED WITH | CORRESPONDING TO $\underline{B}$.

CONSIDER $\quad z = \underline{c}^T \underline{x} = \underline{c}_B^T \underline{x}_B + \underline{c}_N^T \underline{x}_N$ WHERE

$\underline{c} = \begin{pmatrix} \underline{c}_B \\ \underline{c}_N \end{pmatrix}$ WITH $\underline{c}_B \in \mathbb{R}^m$, $\underline{c}_N \in \mathbb{R}^{n-m}$.

FROM (*), $\quad \underline{x}_B = \underline{B}^{-1}\underline{b} - \underline{B}^{-1}\underline{N}\,\underline{x}_N$

D.

SO THAT $z = \underline{c}^T \underline{x} = \underline{c}_B^T (\underline{B}^{-1}\underline{b} - \underline{B}^{-1}\underline{N}\,\underline{x}_N) + \underline{c}_N^T \underline{x}_N$

$$= \underline{c}_B^T \underline{B}^{-1}\underline{b} + (\underline{c}_N^T - \underline{c}_B^T \underline{B}^{-1}\underline{N})\,\underline{x}_N \qquad (**)$$

(*), (**) TELL US ALL WE NEED TO KNOW WHEN COMBINED WITH THE FUNDAMENTAL THEOREM.

LET $\underline{r}^T = \underline{c}_N^T - \underline{c}_B^T \underline{B}^{-1}\underline{N}$ THEN $\underline{r} = (r_1, \dots, r_{n-m})$ IS THE VECTOR OF REDUCED COSTS.

FOR ANY $\underline{x} \in F_{(c)}$, $z = \underline{c}_B^T \underline{B}^{-1}\underline{b} + \sum_{j=1}^{n-m} r_j x_{m+j}$

IF $\underline{x}^* = \begin{pmatrix} \underline{B}^{-1}\underline{b} \\ \underline{0}_{n-m} \end{pmatrix}$ AND $\underline{B}^{-1}\underline{b} \geq \underline{0}_m$ SO THAT $\underline{x}^* \in F_{(c)}$

THEN $z = z(\underline{x}^*) + \sum_{j=1}^{n-m} r_j x_{m+j}$

IF $r_j > 0$ THEN $x_{m+j} > 0$ WILL INCREASE z FROM $z(\underline{x}^*)$ AND WE CAN'T CONCLUDE $\underline{x}^*$ IS OPTIMAL: WE CAN (POSSIBLY) IMPROVE z BY CONSIDERING A BFS INCLUDING x_{m+j}.

[IF $r_j \leq 0$ FOR ALL $j = 1, \dots, n-m$ THEN $\underline{x}^*$ IS AN OPTIMAL SOLUTION].