19|10|2015. LECTURE 8: CB1.11. Ⓐ

LAST TIME: $\underline{x} = \begin{pmatrix} \underline{x}_B \\ \underline{0}_{n-m} \end{pmatrix}$, $\underline{v} = \begin{pmatrix} \underline{v}_B \\ \underline{0}_{n-m} \end{pmatrix}$, $\underline{w} = \begin{pmatrix} \underline{w}_B \\ \underline{0}_{n-m} \end{pmatrix}$

NOW, $\underline{b} = \underline{A}\,\underline{v} = (\underline{B} | \underline{N}) \begin{pmatrix} \underline{v}_B \\ \underline{0}_{n-m} \end{pmatrix} = \underline{B}\,\underline{v}_B$

THUS, AS $\underline{x}, \underline{v} \in F_{(c)}$, $\underline{b} = \underline{B}\,\underline{x}_B = \underline{B}\,\underline{v}_B$ SO THAT

$$\underline{B}(\underline{x}_B - \underline{v}_B) = \underline{0}_m \quad \text{i.e.} \quad \sum_{j=1}^{m} (x_j - v_j)\underline{a}_j = \underline{0}_m .$$

BUT $\underline{a}_1, \ldots, \underline{a}_m$ ARE LINEARLY INDEPENDENT SO $x_j - v_j = 0$ $\forall\, j = 1, \ldots, m$.

i.e. $\underline{x}_B = \underline{v}_B \Rightarrow \underline{x} = \underline{v}$ AND HENCE $\underline{v} = \underline{w}$

THUS, $\underline{v}, \underline{w}$ ARE NOT DISTINCT: CONTRADICTION. $\square$

THEOREM.

CONSIDER A LP PROBLEM IN CANONICAL FORM. SUPPOSE $\underline{x}$ IS AN EXTREME POINT OF $F_{(c)}$ THEN $\underline{x}$ IS A BASIC FEASIBLE SOLUTION.

PROOF.

WE NEED TO SHOW THAT THE COLUMNS OF $\underline{A}$ FOR WHICH $x_j > 0$ ARE LINEARLY INDEPENDENT.

WLOG ASSUME FIRST k COMPONENTS OF $\underline{x}$ ARE NON-ZERO.

$$\text{i.e. } x_j > 0 \quad j = 1, \ldots, k \qquad x_j = 0 \quad j = k+1, \ldots, n.$$

$$\underline{A}\,\underline{x} = \sum_{j=1}^{n} x_j \underline{a}_j = \underline{\sum_{j=1}^{k} x_j \underline{a}_j = \underline{b}} \qquad (*)$$

** ASSUME THAT $\underline{a}_1, \ldots, \underline{a}_k$ ARE LINEARLY DEPENDENT **

(B).

[AIM: PROOF BY CONTRADICTION. WE'RE GOING TO FIND TWO POINTS IN $F_{(c)}$ WHICH $\underline{x}$ LIES ON A SEGMENT BETWEEN SO THAT $\underline{x}$ IS NOT EXTREME].

IN WHICH CASE, THERE EXIST CONSTANTS α_j NOT ALL EQUAL TO ZERO SUCH THAT

$$\sum_{j=1}^{k} \alpha_j \underline{a}_j = \underline{0}_m \qquad (**)$$

[ADDING MULTIPLES OF (**) TO (*) CAN SATISFY $\underline{A}\,\underline{x}' = \underline{b}$. WANT TO MAINTAIN $\underline{x}' \geq \underline{0}_n$].

LET $\mu > 0$ BE SUCH THAT $\mu|\alpha_j| \leq x_j$ FOR ALL $j = 1, \dots, k$.
HENCE,

$$x_j - \mu\alpha_j \geq x_j - \mu|\alpha_j| \geq 0 \quad \left[\begin{array}{l} x_j - \mu|\alpha_j| \text{ if } \alpha_j > 0 \\ x_j + \mu|\alpha_j| \text{ if } \alpha_j < 0 \end{array}\right]$$

$$x_j + \mu\alpha_j \geq x_j - \mu|\alpha_j| \geq 0 \quad \left[\begin{array}{l} x_j + \mu|\alpha_j| \text{ if } \alpha_j > 0 \\ x_j - \mu|\alpha_j| \text{ if } \alpha_j < 0 \end{array}\right]$$

FROM (*) AND (**)

$$\sum_{j=1}^{k} (x_j - \mu\alpha_j)\underline{a}_j = \underline{b}, \quad \sum_{j=1}^{k} (x_j + \mu\alpha_j)\underline{a}_j = \underline{b}.$$

LET $\underline{\alpha} = \overbrace{(\alpha_1, \dots, \alpha_k, 0, \dots, 0)}^{\text{NOTE: } \underline{\alpha} \neq \underline{0}_n}$ THEN $\underline{x} - \mu\underline{\alpha} \in F_{(c)}$ AND $\underline{x} + \mu\underline{\alpha} \in F_{(c)}$. AS $F_{(c)}$ IS CONVEX THEN FOR ALL $\lambda \in [0,1]$

$$\lambda(\underline{x} - \mu\underline{\alpha}) + (1-\lambda)(\underline{x} + \mu\underline{\alpha}) \in F_{(c)}$$

TAKING $\lambda = 1/2$ GIVES $\underline{x} = \frac{1}{2}(\underline{x} - \mu\underline{\alpha}) + \frac{1}{2}(\underline{x} + \mu\underline{\alpha})$

SO THAT $\underline{x}$ IS NOT AN EXTREME POINT : A CONTRADICTION.
[AS NOT ALL $\alpha_j = 0$ IN DEPENDENCE ASSUMPTION, $\mu > 0$
AND $\underline{\alpha} \neq \underline{0}$] . □

NOTES.

1. IF $k < m$ THEN $\underline{x}$ IS A DEGENERATE BASIC SOLUTION.
2. WE CAN FIND $m-k$ COLUMNS FROM $\underline{a}_{k+1}, \dots, \underline{a}_n$ SUCH THAT THESE AND $\underline{a}_1, \dots, \underline{a}_k$ FORM A LINEARLY INDEPENDENT SET : SEE Q3 OF QUESTION SHEET THREE

[DEGENERATE EXAMPLE THIS MORNING $k = 1$ $\{\underline{a}_1\}$ $m-2$
SELECT EITHER $\underline{a}_2$ OR $\underline{a}_3$ TO FORM LINEARLY INDEPENDENT SET].

3.2 THE FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING.

THEOREM.

CONSIDER A LP PROBLEM IN CANONICAL FORM WITH FEASIBLE REGION $F_{(c)}$.

① IF THERE IS A FEASIBLE SOLUTION, THERE IS A BASIC FEASIBLE SOLUTION.

② IF THERE IS AN OPTIMAL SOLUTION, THERE IS AN OPTIMAL BASIC FEASIBLE SOLUTION.

PROOF.

WE SHALL PROVE THE RELATIONSHIP BETWEEN OPTIMAL SOLUTIONS, FEASIBLE SOLUTIONS FOLLOW IN A SIMILAR WAY.

(D).

LET $\underline{x}$ BE AN OPTIMAL FEASIBLE SOLUTION. WLOG LET NON-ZERO COMPONENTS OF $\underline{x}$ BE THE FIRST k COMPONENTS

i.e. $$x_j = \begin{cases} > 0 & j=1,\dots,k \\ 0 & j=k+1,\dots,n. \end{cases}$$

WE HAVE $\sum_{j=1}^{k} x_j \underline{a}_j = \underline{b}$. IF $\underline{a}_1,\dots,\underline{a}_k$ ARE LINEARLY INDEPENDENT WE'RE DONE. OTHERWISE, SUPPOSE $\underline{a}_1,\dots,\underline{a}_k$ LINEARLY DEPENDENT. THEN THERE EXIST CONSTANTS α_j NOT ALL EQUAL TO ZERO SUCH THAT

$$\sum_{j=1}^{k} \alpha_j \underline{a}_j = \underline{0}_m$$

[WE'RE GOING TO SHOW THAT AS LONG AS THE $\underline{a}_1,\dots,\underline{a}_k$ ARE LINEARLY DEPENDENT WE CAN FIND AN OPTIMAL SOLUTION THAT HAS ONE OF THE $x_1,\dots,x_k$ EQUAL TO ZERO AND CONTINUE UNTIL WE HAVE INDEPENDENCE]

LET $\mu > 0$ BE SUCH THAT $\mu|\alpha_j| \le x_j$ FOR ALL $j=1,\dots,k$
THEN (AS IN THE PREVIOUS PROOF)

$$\underline{x} - \mu\underline{\alpha},\ \underline{x} + \mu\underline{\alpha} \in F_{(c)} \text{ WHERE } \underline{\alpha} = (\alpha_1,\dots,\alpha_k,0,\dots,0)^T$$

AS $\underline{x}$ OPTIMAL

$$\underline{c}^T\underline{x} \ge \underline{c}^T(\underline{x} - \mu\underline{\alpha}) = \underline{c}^T\underline{x} - \mu\underline{c}^T\underline{\alpha}$$

i.e. $\mu \underline{c}^T\underline{\alpha} \ge 0$. (*)