19/10/2015. LECTURE 7: 3WN2.1.

EXAMPLE OF BASIC SOLUTIONS CONTINUED.

$$\underline{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

② SELECT $\{x_1, x_3\}$ AS BASIC, x_2 NON-BASIC.

$$\underline{B} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \underline{B}^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{B}^{-1}\underline{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

THE CORRESPONDING BASIC SOLUTION IS $\underline{x} = (-1, 0, 2)^T$ WHICH IS NOT FEASIBLE.

③ SELECT $\{x_2, x_3\}$ AS BASIC, x_1 NON-BASIC.

$$\underline{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \underline{B}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \underline{B}^{-1}\underline{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

THE CORRESPONDING BASIC SOLUTION IS $\underline{x} = (0, 1, 1)^T$ WHICH IS FEASIBLE AND THUS A BASIC FEASIBLE SOLUTION (BFS).

EXAMPLE.

SUPPOSE WE CHANGE $\underline{b}$ TO $\underline{b} = (0, 1)^T$.

①. SELECT $\{x_1, x_2\}$ BASIC, x_3 NON-BASIC.

$$\underline{B}^{-1}\underline{b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \boxed{0} \end{pmatrix}$$

BASIC SOLUTION IS $\underline{x} = (1, \boxed{0}, 0)^T$ WHICH IS FEASIBLE.

NOTE BASIC VARIABLE EQUAL TO ZERO.

(2). SELECT $\{x_1, x_3\}$ BASIC, x_2 NON-BASIC.

$$\underline{B}^{-1}\underline{b} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

BASIC SOLUTION IS $\underline{x} = (1, 0, \boxed{0})^T$ WHICH IS FEASIBLE.

NOTE BASIC VARIABLE EQUAL TO ZERO.

** TWO BASIS GIVE THE SAME SOLUTION **

(3). SELECT $\{x_2, x_3\}$ BASIC, x_1 NON-BASIC

$$\underline{B}^{-1}\underline{b} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

BASIC SOLUTION IS $\underline{x} = (0, -1, 1)^T$ WHICH IS NOT FEASIBLE.

DEFINITION.

A DEGENERATE BASIC SOLUTION OCCURS WHEN ONE OR MORE OF THE BASIC VARIABLES IN A BASIC SOLUTION HAS A VALUE OF ZERO.

MOTIVATING EXAMPLE THUS FAR HAVE SHOWN THAT IF A SOLUTION EXISTS, IT CAN BE FOUND | OBTAINED AT AN EXTREME POINT.

EXAMPLE IN THE PROBLEMS CLASS SUGGESTED A ONE-TO-ONE CORRESPONDENCE BETWEEN EXTREME POINTS AND BASIC SOLUTIONS. [SHOW CORRESPONDENCE BETWEEN EXTREME POINTS IN STANDARD | CANONICAL PROBLEMS IN Q5 OF QUESTION SHEET THREE). WE'RE NOW GOING TO PROVE THESE RESULTS.

(C).

DEFINITION

SUPPOSE S IS A CONVEX SET. A POINT $\underline{x} \in S$ IS SAID TO BE AN EXTREME POINT OF S IF THERE ARE NO DISTINCT POINTS $\underline{v}, \underline{w} \in S$ SUCH THAT, FOR SOME $\lambda \in (0,1)$, $\underline{x} = \lambda \underline{v} + (1-\lambda)\underline{w}$.

i.e. $\underline{x}$ EXTREME POINT IF IT DOESN'T LIE ON A LINE SEGMENT CONNECTING TWO OTHER POINTS.

NOTE: WE'VE SHOWN FEASIBLE REGIONS OF LP PROBLEMS ARE CONVEX SETS.

THEOREM.

CONSIDER A LP PROBLEM IN CANONICAL FORM WITH FEASIBLE REGION $F_{(C)} = \{\underline{x} \in \mathbb{R}^n : \underline{A}\,\underline{x} = \underline{b}, \underline{x} \geq \underline{0}_n\}$. SUPPOSE $\underline{x}$ IS A BASIC FEASIBLE SOLUTION. THEN $\underline{x}$ IS AN EXTREME POINT OF $F_{(C)}$.

PROOF.

IDEA: PROOF BY CONTRADICTION. $\underline{x}$ BASIC FEASIBLE SOLUTION SO COLUMNS OF $\underline{A}$ CORRESPONDING TO VARIABLE IN BASIS OF LINEARLY INDEPENDENT i.e. $\sum_{j=1}^{m} \alpha_j \underline{a}_j = \underline{0}_m \Rightarrow \alpha_j = 0 \ \forall j = 1, \dots, m$.
WE'LL ASSUME $\underline{x}$ IS NOT AN EXTREME POINT AND USE LINEAR INDEPENDENCE TO SHOW NO DISTINCT $\underline{v}, \underline{w}$.

LET $\underline{x}$ BE A BFS. WLOG, $\underline{x} = \begin{pmatrix} \underline{x}_B \\ \underline{0}_{n-m} \end{pmatrix}$ WITH $\underline{A} = (\underline{B} \mid \underline{N})$

WHERE $\underline{B}$ IS AN INVERTIBLE $m \times m$ MATRIX.

i.e. $\underline{B} = (\underline{a}_1, \dots \underline{a}_m)$ HAS m LINEARLY INDEPENDENT COLUMNS WITH $\underline{B}\,\underline{x}_B = \underline{b}$

6.

** SUPPOSE THAT $\underline{x}$ IS NOT AN EXTREME POINT **

$\Rightarrow \exists \ \underline{v} = (v_1, \dots, v_n)^T, \underline{w} = (w_1, \dots w_n)^T \in F_{(1)}$ SUCH THAT $\underline{x} = \lambda \underline{v} + (1-\lambda)\underline{w}$ WITH $\underline{v}, \underline{w}$ DISTINCT

NOW, $x_j = 0 \ \forall \ j = m+1, \dots, n$

$$\Rightarrow \quad 0 = x_j = \lambda v_j + (1-\lambda) w_j \quad \forall \ j = m+1, \dots, n.$$

AS $\underline{v}, \underline{w} \in F_{(1)}$ THEN $v_j, w_j \geq 0$ AND $\lambda \in (0,1)$. THUS,

$$x_j = v_j = w_j = 0 \ \forall \ j = m+1, \dots, n.$$

THUS, $\underline{v} = \begin{pmatrix} \underline{v}_B \\ \underline{0}_{n-m} \end{pmatrix}$ AND $\underline{w} = \begin{pmatrix} \underline{w}_B \\ \underline{0}_{n-m} \end{pmatrix}$ WHERE $\underline{v}_B, \underline{w}_B \in \mathbb{R}^m$.

NOW, $\underline{b} = \underline{A}\underline{v} = (\underline{B} \,|\, \underline{N}) \begin{pmatrix} \underline{v}_B \\ \underline{0}_{n-m} \end{pmatrix} = \underline{B}\,\underline{v}_B$

THUS, $\underline{B}\,\underline{x}_B = \underline{B}\,\underline{v}_B \Rightarrow \underline{B}\,(\underline{x}_B - \underline{v}_B) = \underline{0}_m$

$$\Rightarrow \sum_{j=1}^{m} (x_j - v_j)\, \underline{a}_j = \underline{0}_m.$$

BUT $\underline{a}_1, \dots, \underline{a}_m$ ARE LINEARLY INDEPENDENT $\Rightarrow x_j - v_j = 0 \ \forall \ j = 1, \dots, m.$

i.e. $\underline{x}_B = \underline{v}_B \Rightarrow \underline{x} = \underline{v}$ AND HENCE $\underline{v} = \underline{w}$.

THUS, $\underline{v}, \underline{w}$ ARE NOT DISTINCT : CONTRADICTION.

$\square$