12|10|2015. LECTURE 6: CB1.11. (A)

3. THE FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING.

3.1 BASIC SOLUTIONS.

WE CONSIDER SOLVING A LP PROBLEM IN CANONICAL FORM

i.e. FIND AN $\underline{x} \in \mathbb{R}^n$ THAT WILL

$$\begin{aligned} \text{maximise} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & \underline{A}\,\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{aligned}$$

WHERE $\underline{A} \in \mathbb{R}^{m \times n}$, $\underline{c} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^m$. LET $F_{(c)}$ DENOTE THE SET OF FEASIBLE SOLUTIONS.

CONSIDER THE MATRIX $\underline{A}$. RECALL THAT:

COLUMN RANK: LARGEST SUBSET OF COLUMNS OF $\underline{A}$ WHICH ARE LINEARLY INDEPENDENT.
ROW RANK: LARGEST SUBSET OF ROWS OF $\underline{A}$ WHICH ARE LINEARLY INDEPENDENT.

[VECTORS $\underline{u}_1, \ldots, \underline{u}_k$ ARE LINEARLY INDEPENDENT IF THEY CANNOT BE EXPRESSED IN THE FORM $\sum_{j=1}^{k} \alpha_j \underline{u}_k = \underline{0}$ WHERE CONSTANTS $\alpha_1, \ldots, \alpha_k$ ARE NOT ALL EQUAL TO ZERO].

$$\text{RANK} = \overbrace{\text{ROW RANK}}^{\leq m} = \overbrace{\text{COLUMN RANK}}^{\leq n}$$

SO, $\text{RANK}(\underline{A}) \leq \min(m, n)$.

ASSUMPTION.

FOR A LP PROBLEM IN CANONICAL FORM, WE SHALL ASSUME $m \leq n$ WITH RANK($\underline{A}$)$=m$.

- ASSUME AT LEAST AS MANY VARIABLES AS EQUALITY CONSTRAINTS.
- ROWS OF $\underline{A}$ ARE LINEARLY INDEPENDENT CORRESPONDING TO LINEAR INDEPENDENCE OF m EQUATIONS.
- A LINEAR DEPENDENCE WOULD MEAN EITHER NO SOLUTIONS OR REDUNDANT CONSTRAINTS.

$$\text{e.g.}\quad \begin{aligned} x_1 + x_2 &= 5 \\ x_2 + x_3 &= 2 \\ x_1 + 2x_2 + x_3 &= 7 \leftarrow \text{REDUNDANT} \end{aligned}$$

RECALL THE COLUMN REPRESENTATION OF $\underline{A}\,\underline{x}$

$$\underline{A}\,\underline{x} = \sum_{j=1}^{n} x_j \underline{a}_j \qquad [\,\underline{A} = (\underline{a}_1 \ldots \underline{a}_n)\,]$$

IF RANK($\underline{A}$)$=m$ THERE IS A SET OF m OF THE COLUMNS OF $\underline{A}$ WHICH ARE LINEARLY INDEPENDENT. FOR SIMPLICITY OF NOTATION, ASSUME (WLOG) FIRST m COLUMNS ARE LINEARLY INDEPENDENT.

$$\underline{A}\,\underline{x} = \underbrace{\sum_{j=1}^{m} x_j \underline{a}_j}_{\underline{a}_1,\ldots,\underline{a}_m \text{ LINEARLY INDEPENDENT}} + \sum_{j=m+1}^{n} x_j \underline{a}_j = \underline{B}\,\underline{x}_B + \underline{N}\,\underline{x}_N$$

WHERE $\underline{B} = (\underline{a}_1 \ldots \underline{a}_m) \in \mathbb{R}^{m\times m}$

$\underline{N} = (\underline{a}_{m+1} \ldots \underline{a}_n) \in \mathbb{R}^{m\times(n-m)}$

$\underline{x}_B = (x_1,\ldots,x_m)^T \in \mathbb{R}^m$

$\underline{x}_N = (x_{m+1},\ldots,x_n)^T \in \mathbb{R}^{n-m}$

NOTE: $\underline{A}\,\underline{x} = \underline{B}\,\underline{x}_B + \underline{N}\,\underline{x}_N = (\underline{B} \mid \underline{N}) \begin{pmatrix} \underline{x}_B \\ \underline{x}_N \end{pmatrix}$

$\underline{B}$ $m \times m$ MATRIX WITH m LINEARLY INDEPENDENT COLUMNS. THUS, RANK$(\underline{B}) = m$ SO THAT $\underline{B}$ IS INVERTIBLE.

THUS, $\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{B}^{-1}\underline{A}\,\underline{x} = \underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b}$

TAKING $\underline{x}_N = (0, \ldots, 0)^T$ LEADS TO $\boxed{\underline{x} = \begin{pmatrix} \underline{B}^{-1}\underline{b} \\ \underline{0}_{n-m} \end{pmatrix}}$ IS A SOLUTION TO $\underline{A}\,\underline{x} = \underline{b}$.

FOR $\underline{x} \in F_{(c)}$ WE ALSO NEED $\underline{x} \geq \underline{0}_n$ i.e. $\boxed{\underline{B}^{-1}\underline{b} \geq \underline{0}_m}$

DEFINITION.

ANY COLLECTION $\{x_{(1)}, \ldots, x_{(m)}\}$ OF m OF THE n VARIABLES $x_1, \ldots, x_n$ FOR WHICH THE CORRESPONDING COLUMNS $\{a_{(1)}, \ldots, a_{(m)}\}$ OF THE MATRIX $\underline{A}$ ARE LINEARLY INDEPENDENT ARE A BASIS. A VARIABLE IN THE BASIS IS BASIC WHILST A VARIABLE NOT IN THE BASIS IS NON-BASIC. A VECTOR $\underline{x}$ SUCH THAT $\underline{A}\,\underline{x} = \underline{b}$ WITH ALL OF THE $n-m$ NON-BASIC VARIABLES EQUAL TO 0 IS A BASIC SOLUTION. IF, IN ADDITION, $\underline{x} \geq \underline{0}_n$ THEN $\underline{x}$ IS A BASIC FEASIBLE SOLUTION.

EXAMPLE.

CONSIDER THE CONSTRAINT $\underline{A}\,\underline{x} = \underline{b}$ WHERE

$$\underline{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ WITH } \underline{x} \geq \underline{0}_3 .$$

(D)

RANK $(\underline{A}) = 2$. THERE ARE AT MOST $\binom{3}{2} = 3$ BASIC SOLUTIONS. IN THIS CASE, EACH PAIR OF COLUMNS IS LINEARLY INDEPENDENCE SO THERE ARE THREE BASIC SOLUTIONS.

①. SELECT $\{x_1, x_2\}$ AS BASIS, x_3 NON-BASIC.

$$\underline{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{B}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{B}^{-1}\underline{b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

BASIC SOLUTION IS $\underline{x} = (1, 2, 0)^T$ WHICH IS FEASIBLE.

– HERE.

②. SELECT $\{x_1, x_3\}$ AS BASIS, x_2 NON-BASIC.

$$\underline{B} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \underline{B}^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{B}^{-1}\underline{b} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

BASIC SOLUTION IS $\underline{x} = (-1, 0, 2)^T$ WHICH IS NOT FEASIBLE.

③. SELECT $\{x_2, x_3\}$ AS BASIS, x_1 NON-BASIC.

$$\underline{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \underline{B}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \underline{B}^{-1}\underline{b} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

BASIC SOLUTION IS $\underline{x} = (0, 1, 1)^T$ WHICH IS FEASIBLE.