12|10|2015. LECTURE 5 : 3WN2.1.

DEFINITION.

CORRESPONDING TO THE HYPERPLANE $H = \{\underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} = d\}$ ARE THE:

POSITIVE CLOSED HALF-SPACE $H_+ = \{\underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} \geq d\}$

NEGATIVE CLOSED HALF-SPACE $H_- = \{\underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} \leq d\}$

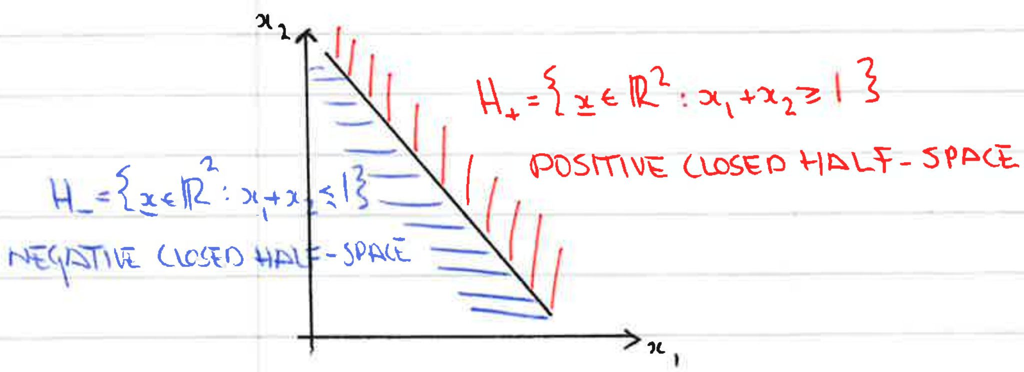
POSITIVE OPEN HALF-SPACE $\tilde{H}_+ = \{\underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} > d\}$

NEGATIVE OPEN HALF-SPACE $\tilde{H}_- = \{\underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} < d\}$

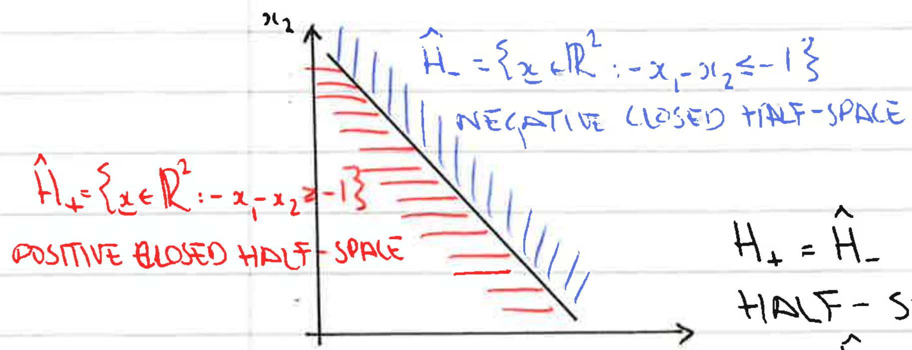
NOTE THAT $H = H_+ \cap H_-$

EXAMPLE

THE LINE $x_1 + x_2 = 1$ IS A HYPERPLANE IN $\mathbb{R}^2$.



NOTE: $x_1 + x_2 \geq 1 \iff -x_1 - x_2 \leq -1$. LINE $-x_1 - x_2 = -1$ IS A HYPERPLANE, $\hat{H}$ SAY, IN $\mathbb{R}^2$.



$H_+ = \hat{H}_-$ IS THE NEGATIVE CLOSED HALF-SPACE OF THE HYPERPLANE $\hat{H} = \{\underline{x} \in \mathbb{R}^2 : -x_1 - x_2 = -1\}$

EXAMPLE.

THE iTH CONSTRAINT OF A LP PROBLEM IN STANDARD FORM IS THE NEGATIVE CLOSED HALF-SPACE OF THE HYPERPLANE

$$H_i = \{ \underline{x} \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij} x_j = b_i \}$$

PROPOSITION.

THE NEGATIVE CLOSED HALF-SPACE $H_- = \{ \underline{x} \in \mathbb{R}^n : \underline{c}^T \underline{x} \leq d \}$ AND THE POSITIVE CLOSED HALF-SPACE $H_+ = \{ \underline{x} \in \mathbb{R}^n : \underline{c}^T \underline{x} \geq d \}$ ARE BOTH CLOSED CONVEX SETS.

[NOTE: H_-, H_+ CLOSED CONVEX SET $\Rightarrow$ $H_- \cap H_+$ CLOSED CONVEX SET i.e. H CLOSED CONVEX SET (SHOWED THIS DIRECTLY LAST TIME)]

DEFINITION.

THE INTERSECTION OF A FINITE NUMBER OF CLOSED HALF-SPACES IS A CLOSED POLYHEDRON

PROPOSITION.

A CLOSED POLYHEDRON IS A CONVEX SET.

PROOF.

CLOSED HALF-SPACE IS CONVEX, INTERSECTION OF CONVEX SETS IS CONVEX.

DEFINITION.

A CLOSED POLYTOPE IS A BOUNDED CLOSED POLYHEDRON.

EXAMPLE.

CONSTRAINTS $\underline{x} \geq \underline{0}_n$ ARE THE INTERSECTION OF n CLOSED HALF-SPACES CORRESPONDING TO

$\underline{e}_i^T \underline{x} \geq 0$ WHERE $\underline{e}_i$ IS THE ith COLUMN OF THE $n \times n$ IDENTITY MATRIX $I_n = (\underline{e}_1 \dots \underline{e}_n)$.

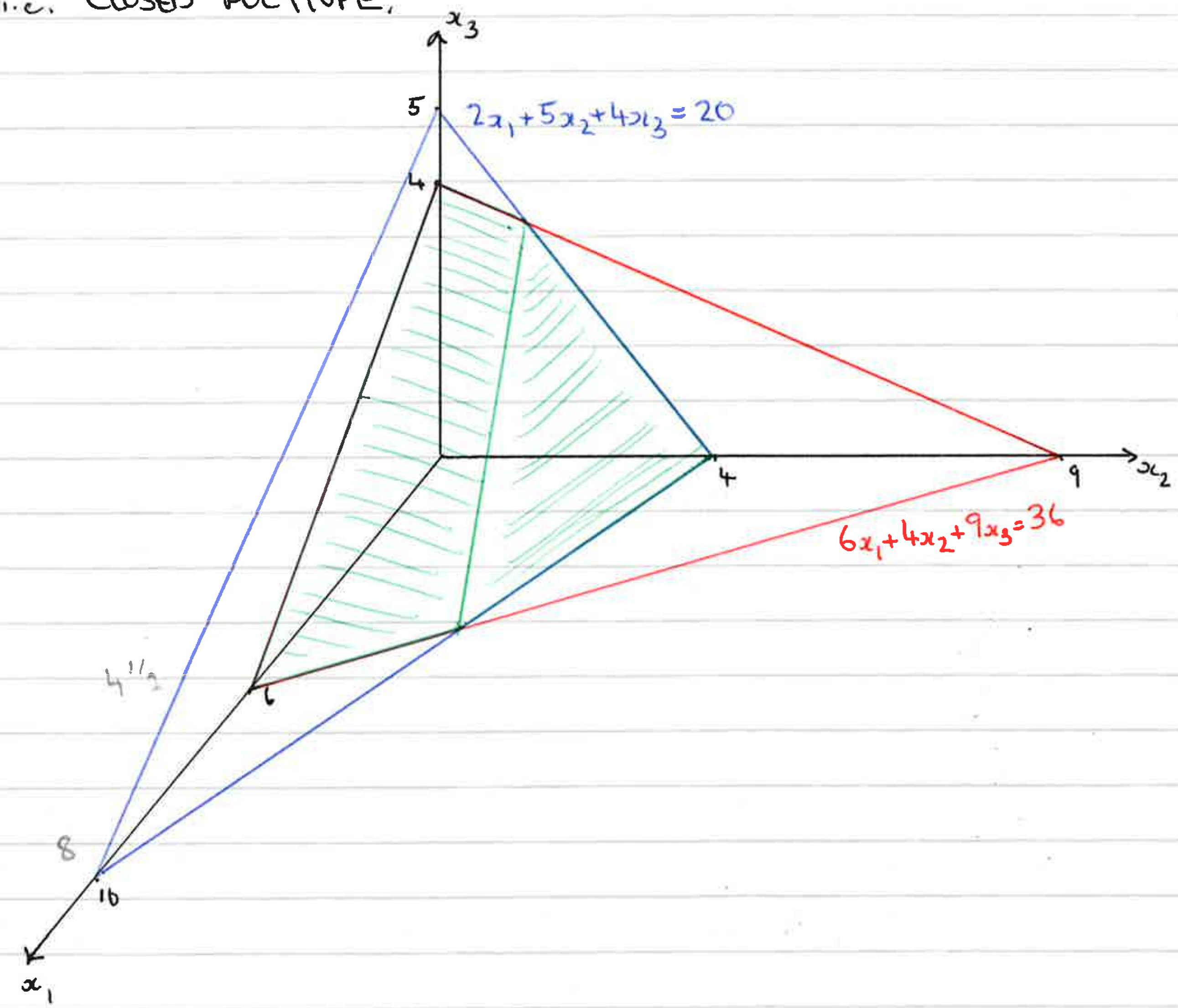
HENCE, $\underline{x} \geq \underline{0}_n$ IS:
- CLOSED POLYHEDRON
- CONVEX
- NOT BOUNDED SO NOT A CLOSED POLYTOPE.

EXAMPLE.

CONSIDER THE SET OF CONSTRAINTS

$$6x_1 + 4x_2 + 9x_3 \leq 36$$
$$2x_1 + 5x_2 + 4x_3 \leq 20$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

THESE FORM A BOUNDED CLOSED CONVEX POLYHEDRON i.e. CLOSED POLYTOPE.



EXAMPLE

$F_{(S)} = \{ \underline{x} \in \mathbb{R}^n : \underline{A}\,\underline{x} \le \underline{b},\ \underline{x} \ge \underline{0}_n \}$ FEASIBLE SET OF SOLUTIONS FOR LP IN STANDARD FORM, IS A CLOSED CONVEX POLYHEDRON.

$$\underbrace{\begin{array}{rl} \underline{A}\,\underline{x} \le \underline{b} & \text{INTERSECTION OF } m \text{ NEGATIVE H-S} \\ \underline{x} \ge \underline{0}_n & \text{INTERSECTION OF } n \text{ POSITIVE H-S} \end{array}}_{\text{INTERSECTION OF } n+m \text{ CLOSED H-S}}$$

EXAMPLE

$F_{(C)} = \{ \underline{x} \in \mathbb{R}^n : \underline{A}\,\underline{x} = \underline{b},\ \underline{x} \ge \underline{0}_n \}$ FEASIBLE SET OF SOLUTIONS FOR LP IN CANONICAL FORM IS A CLOSED CONVEX POLYHEDRON.

$$\underbrace{\begin{array}{rl} \underline{A}\,\underline{x} = \underline{b} & \text{INTERSECTION OF } 2m \text{ H-S} \left(\begin{array}{l} \underline{A}\,\underline{x} \le \underline{b} \\ \underline{A}\,\underline{x} \ge \underline{b} \end{array} \right) \\ \underline{x} \ge \underline{0}_n & \text{INTERSECTION OF } n \text{ POSITIVE H-S} \end{array}}_{\text{INTERSECTION OF } 2m+n \text{ H-S.}}$$

– HERE.

3. THE FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING

3.1 BASIC SOLUTIONS.

WE CONSIDER SOLVING A LP PROBLEM IN CANONICAL FORM

i.e. FIND $\underline{x} \in \mathbb{R}^n$ THAT WILL

$$\begin{array}{ll} \text{maximum} & z = \underline{c}^T \underline{x} \\ \text{subject to} & \underline{A}\,\underline{x} = \underline{b} \\ & \underline{x} \ge \underline{0}_n \end{array}$$

WHERE $\underline{A} \in \mathbb{R}^{m \times n}$, $\underline{c} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^m$. LET $F_{(C)}$ DENOTE THE SET OF FEASIBLE SOLUTIONS.