5/10/2015. LECTURE 4: CB1.11. Ⓐ

EXAMPLE (COST MINIMISATION EXAMPLE).

IN STANDARD FORM THE PROBLEM IS

$$\begin{aligned} \text{maximum } & z = -40x_1 - 80x_2 \\ \text{subject to } & -10x_1 - 5x_2 \leq -10 \\ & -2x_1 - 3x_2 \leq -3 \\ & -2x_1 - 8x_2 \leq -4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

WE ADD A SLACK VARIABLE TO EACH CONSTRAINT. THE PROBLEM IN CANONICAL FORM IS:

$$\begin{aligned} \text{maximum } & z = -40x_1 - 80x_2 \\ \text{subject to } & -10x_1 - 5x_2 + s_1 = -10 \\ & -2x_1 - 3x_2 + s_2 = -3 \\ & -2x_1 - 8x_2 + s_3 = -4 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

NOTE $x_1 = x_2 = 0$, $s_1 = -10$, $s_2 = -3$, $s_3 = -4$ SOLVES THIS SYSTEM BUT IS NOT FEASIBLE.

EXAMPLE OF A BASIC SOLUTION (AS WE'LL SEE LATER).

SOLVING LP PROBLEMS: THE GRAPHICAL METHOD.

IN PRINCIPLE, IF THE NUMBER OF (NON-SLACK) VARIABLES ≤ 3 WE CAN SOLVE THE PROBLEM GRAPHICALLY.

① CONSTRUCT THE FEASIBLE REGION BY PLOTTING THE CONSTRAINTS. FEASIBLE REGION IS THE INTERSECTION OF THESE.

e.g. CONSTRAINT $a_{i1}x_1 + a_{i2}x_2 \leq b_i$ BOUNDED BY STRAIGHT LINE $a_{i1}x_1 + a_{i2}x_2 = b_i$

$$\left(\text{or} \quad x_2 = \frac{b_i}{a_{i2}} - \frac{a_{i1}}{a_{i2}} x_1 \right)$$

$a_{i1}x_1 + a_{i2}x_2 \le b_i$ IS THEN EITHER THE LINE AND ABOVE OR THE LINE AND BELOW (CHECK THE ORIGIN!)

(2). MAXIMISING $z(\underline{x}) = \underline{c}^T \underline{x}$ CORRESPONDS TO MOVING $z(\underline{x})$ PARALLEL TO ITSELF IN THE DIRECTION $\underline{c}$ AS FAR AS POSSIBLE UNTIL IT REACHES POINT WHERE CANNOT BE INCREASED FURTHER. AT THIS POINT, IT REACHES OPTIMALITY.

e.g. MAXIMUM $z(\underline{x}) = c_1x_1 + c_2x_2$. PLOT LINE $c_1x_1 + c_2x_2 = z$ FOR SOME FIXED z $\left(x_2 = \frac{z}{c_2} - \frac{c_1 x_1}{c_2}\right)$

MOVE LINE IN PARALLEL DIRECTION (c_1, c_2) AS FAR AS POSSIBLE WHILST REMAINING FEASIBLE.

IN PRACTICE, WANT TO SOLVE PROBLEMS WITH MANY THOUSANDS OF VARIABLES. APPROACH EXPLOITS THE GEOMETRY OF LINEAR PROGRAMMING PROBLEMS.

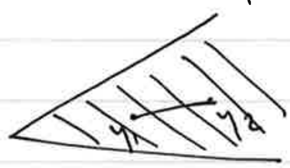
2.3 GEOMETRY OF SYSTEMS OF LINEAR INEQUALITIES.

DEFINITION.

A NON-EMPTY SET $S \subseteq \mathbb{R}^n$ IS A <u>CONVEX SET</u> IF, FOR ALL $\underline{y}_1, \underline{y}_2 \in S$ AND $\lambda \in [0,1]$

$$\lambda \underline{y}_1 + (1-\lambda)\underline{y}_2 \in S.$$

i.e. SET IS CONVEX IF EVERY POINT ON THE LINE SEGMENT JOINING $\underline{y}_1, \underline{y}_2$ IS ALSO A MEMBER OF S.



CONVEX

NOT CONVEX.

[WHERE ARE WE GOING? SHOW THAT FEASIBLE SET OF SOLUTIONS IS CLOSED CONVEX POLYHEDRON].

NOTE THAT IF $y_1, \dots, y_n \in S$ THEN THE CONVEX COMBINATION OF THESE POINTS IS

$$\sum_{j=1}^{n} \lambda_j y_j \quad \text{WHERE} \quad \lambda_j \geq 0 \text{ AND } \sum_{j=1}^{n} \lambda_j = 1.$$

DEFINITION.

A CLOSED CONVEX SET IS A CONVEX SET WHICH INCLUDES ITS BOUNDARIES.

e.g. INTERIOR OF A CIRCLE - CONVEX
INTERIOR OF A CIRCLE AND THE CIRCLE - CLOSED CONVEX

PROPOSITION.

THE INTERSECTION OF ANY COLLECTION OF CONVEX SETS IS CONVEX.

PROOF.

QUESTION SHEET ONE.

DEFINITION.

LET $\underline{n} \in \mathbb{R}^n$ AND $d \in \mathbb{R}$ AND ASSUME $\underline{n} \neq \underline{0}_n$. THE SET

$$H = \{ \underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} = d \}$$

IS CALLED A HYPERPLANE WITH NORMAL VECTOR $\underline{n}$.

EXAMPLE

IN $\mathbb{R}$, A HYPERPLANE IS $n x_1 = d$ i.e. A POINT
IN $\mathbb{R}^2$, A HYPERPLANE IS $n_1 x_1 + n_2 x_2 = d$ i.e. A LINE
IN $\mathbb{R}^3$, A HYPERPLANE IS $n_1 x_1 + n_2 x_2 + n_3 x_3 = d$ i.e. A PLANE

IN A LP PROBLEM, OBJECTIVE FUNCTION $z = \underline{c}^T \underline{x}$ IS A HYPERPLANE SOMETIMES CALLED THE <u>OBJECTIVE HYPERPLANE</u>. IN A LP PROBLEM IN CANONICAL FORM, EACH CONSTRAINT

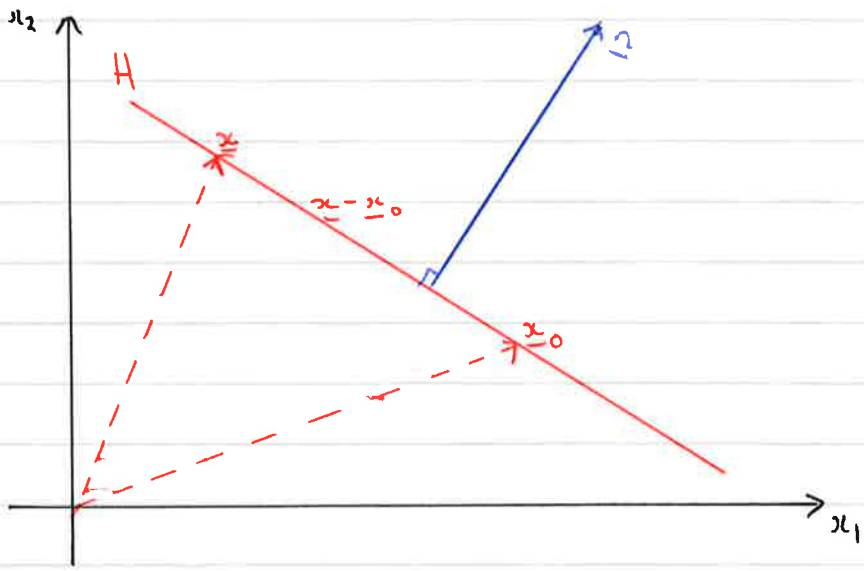
$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{IS A HYPERPLANE.}$$

NOTE IF $\underline{x}_0$ IS ANY FIXED POINT IN H THEN $\underline{c}^T \underline{x}_0 = d$.

ANY $\underline{x} \in H$ THEN SATISFIES

$$\underline{n}^T (\underline{x} - \underline{x}_0) = 0$$

i.e. $\underline{n}$ IS ORTHOGONAL (OR NORMAL) TO $\underline{x} - \underline{x}_0$ FOR EACH $\underline{x} \in H$.



PROPOSITION.

A HYPERPLANE IS A ((CLOSED) CONVEX SET.

PROOF.

SUPPOSE $\underline{x}_1, \underline{x}_2 \in H$ THEN, FOR ANY $\lambda \in [0,1]$

$$\underline{n}^T (\lambda \underline{x} + (1-\lambda)\underline{x}_2) = \lambda \underline{n}^T \underline{x}_1 + (1-\lambda)\underline{n}^T \underline{x}_2$$
$$= \lambda d + (1-\lambda) d = d.$$

SO, $\lambda \underline{x}_1 + (1-\lambda) \underline{x}_2 \in H$. $\square$

A HYPERPLANE DIVIDES $\mathbb{R}^n$ INTO TWO REGIONS, OR HALF-SPACES.

— HERE.

DEFINITION.

CORRESPONDING TO THE HYPERPLANE $H = \{ \underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} = d \}$ ARE THE POSITIVE CLOSED HALF-SPACE

$$H_+ = \{ \underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} \geq d \}$$

THE NEGATIVE CLOSED HALF-SPACE

$$H_- = \{ \underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} \leq d \}$$

AND THE POSITIVE AND NEGATIVE OPEN HALF-SPACES

$$\tilde{H}_+ = \{ \underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} > d \}$$
$$\tilde{H}_- = \{ \underline{x} \in \mathbb{R}^n : \underline{n}^T \underline{x} < d \}$$

NOTE THAT $H = H_+ \cap H_-$