(A)

5/10/2015. LECTURE 3: 3WN2.1.

ANY LINEAR PROGRAMMING PROBLEM CAN BE EXPRESSED AS A STANDARD MAXIMISATION PROBLEM.

① MINIMISING $\underline{c}^T\underline{x}$ EQUIVALENT TO MAXIMISING $-\underline{c}^T\underline{x}$

② CONSTRAINT OF THE FORM $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ IS EQUIVALENT TO $-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$

③ CONSTRAINT OF THE FORM $\sum_{j=1}^{n} a_{ij}x_j = b_i$ IS EQUIVALENT TO THE PAIR $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ AND $-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$

[WE TALKED ABOUT HOW TO REDEFINE THE VARIABLES TO ENSURE THAT $\underline{x} \geq \underline{0}_n$].

IN THIS COURSE, WE WILL WORK WITH MAXIMISATION

"STANDARD MAXIMISATION PROBLEM" ≡ "STANDARD PROBLEM"

EXAMPLE (COST MINIMISATION PROBLEM).

minimise $40x_1 + 80x_2$
subject to
$$10x_1 + 5x_2 \geq 10$$
$$2x_1 + 3x_2 \geq 3$$
$$2x_1 + 8x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

$\Leftrightarrow$

maximise $-40x_1 - 80x_2$
subject to
$$-10x_1 - 5x_2 \leq -10$$
$$-2x_1 - 3x_2 \leq -3$$
$$-2x_1 - 8x_2 \leq -4$$
$$x_1, x_2 \geq 0$$

i.e. $\underline{A} = \begin{pmatrix} -10 & -5 \\ -2 & -3 \\ -2 & -8 \end{pmatrix}$, $\underline{c} = \begin{pmatrix} -40 \\ -80 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} -10 \\ -3 \\ -4 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

IN MATRIX-VECTOR FORM.

(B)

2.2 CANONICAL FRAMEWORK.

DEFINITION.

THE CANONICAL (MAXIMISATION) PROBLEM IS TO FIND A VECTOR $\underline{x} \in \mathbb{R}^n$ THAT WILL:

$$\begin{aligned} \text{maximum} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & \underline{A}\,\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{aligned}$$

WHERE THE MATRIX $\underline{A} \in \mathbb{R}^{m \times n}$, $\underline{b} \in \mathbb{R}^m$, $\underline{c} \in \mathbb{R}^n$ AND $\underline{0}_n$ IS THE n- DIMENSIONAL ZERO VECTOR.

WE NOW SHOW THAT THE STANDARD PROBLEM CAN BE EXPRESSED AS A CANONICAL PROBLEM. TO DO THIS WE INTRODUCE SLACK VARIABLES.

SUPPOSE PROBLEM IS IN STANDARD FORM WITH iTH CONSTRAINT

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$\text{i.e. } b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0 .$$

INTRODUCE $s_i \geq 0$ TO TAKE UP THE SLACK.

$$\Rightarrow \text{SET} \quad s_i = b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0$$

THEN, REARRANGING, $\underline{\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i}$

IF WE DO THIS TO EACH OF THE m CONSTRAINTS, WE INTRODUCE m NEW VARIABLES $\underline{s} = (s_1, \dots, s_m)^T$ WITH $\underline{s} \geq \underline{0}_m$.

IN MATRIX FORM WE HAVE:

$$\underline{A}\,\underline{x} + \underline{I}_m\,\underline{s} = \underline{b}$$

WHERE $\underline{I}_m$ IS THE $m \times m$ IDENTITY MATRIX. LET

$$\underline{A}' = \left.\left(\overbrace{\underline{A}}^{n} \;\middle|\; \overbrace{\underline{I}_m}^{m} \right)\right\} m$$

$m \times (m+n)$ AUGMENTED MATRIX. THEN:

$$\left(\underline{A} \mid \underline{I}_m \right) \begin{pmatrix} \underline{x} \\ \underline{s} \end{pmatrix} = \underline{b}$$

NON-SLACK VARIABLES

LET $\underline{x}' = \begin{pmatrix} \underline{x} \\ \underline{s} \end{pmatrix}$

SLACK VARIABLES

NOTING THAT $z = \underline{c}^T \underline{x} = \underline{c}^T \underline{x} + \underline{0}_m^T \underline{s}_m = \underline{c}'^T \underline{x}'$ WHERE $\underline{c}' = \begin{pmatrix} \underline{c} \\ \underline{0}_m \end{pmatrix}$ WE CAN WRITE THE STANDARD PROBLEM AS:

$$\begin{array}{ll} \text{maximize} & z = \underline{c}'^T \underline{x}' \\ \text{subject to} & \underline{A}'\,\underline{x}' = \underline{b} \\ & \underline{x}' \geq \underline{0}_{n+m} \end{array} \qquad (*)$$

NOTES:

1. CONVERTED A PROBLEM WITH n VARIABLES AND m CONSTRAINTS INTO ONE WITH $n+m$ VARIABLES AND m CONSTRAINTS

   NOTE: $m \leq n+m$

2. STANDARD PROBLEM AND EQUIVALENT CANONICAL PROBLEM HAVE THE SAME OBJECTIVE FUNCTION.

IF $\underline{x}' = \begin{pmatrix} \underline{x} \\ \underline{s} \end{pmatrix}$ SOLVES $(*)$ THEN $\underline{x}$ SOLVES STANDARD PROBLEM:

WHAT WE'LL DO: SOLVE THE CANONICAL PROBLEM AND THE SOLUTION RESTRICTED TO THE NON-SLACK VARIABLES WILL SOLVE THE STANDARD PROBLEM.

ALTERNATIVELY, SOLVING THE STANDARD PROBLEM AND FINDING THE CORRESPONDING SLACK VARIABLES WILL SOLVE THE CANONICAL PROBLEM. WE'LL PROVE THIS CORRESPONDANCE IN THE PROBLEMS CLASS ON THURSDAY.

③. $\underline{A}$ $m \times n$, $\underline{A}'$ $m \times (n+m)$ SO $\underline{A}'$ HAS MORE COLUMNS THAN ROWS. WHEN WE LOOK TO SOLVE CANONICAL PROBLEMS WE'LL ASSUME THAT THERE ARE AT LEAST AS MANY VARIABLES AS CONSTRAINTS AND TYPICALLY MORE.

④. LATER, WE'LL ASSUME PROBLEMS PRESENTED IN CANONICAL FORM WITH n VARIABLES AND m CONSTRAINTS WHERE $n \geq m$ AND WE WON'T EXPLICITLY DISTINGUISH BETWEEN SLACK AND NON-SLACK VARIABLES.

EXAMPLE (COST MINIMISATION EXAMPLE)

IN STANDARD FORM THE PROBLEM IS

maximum $z = -40x_1 - 80x_2$

subject to
$$\begin{aligned} -10x_1 - 5x_2 &\leq -10 \\ -2x_1 - 3x_2 &\leq -3 \\ -2x_1 - 8x_2 &\leq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$$