7/12/2015. LECTURE 22: 3WN2.1. (A)

The solution is not optimal. The $[t_{ij}]$ represent the REDUCED COSTS.

[Solving to find a y with $\underline{c}^T \underline{x} = \underline{b}^T \underline{y}$. $\underline{x}$ is a bfs so $\underline{c}^T \underline{x} = \underline{c}_B^T \underline{B}^{-1} \underline{b}$ so $\underline{y}^T = \underline{c}_B^T \underline{B}^{-1}$ is the dual solution. Q5 QUESTION SHEET SEVEN:

$$\underline{c}_N^T - \underline{c}_B^T \underline{B}^{-1} \underline{N} = \underline{c}_N^T - \underline{y}^T \underline{N} = c_{ij} - u_i - v_j$$

for x_{ij} not in the basis].

Minimisation problem so want the reduced costs to be positive for optimal solution.

MOVING TO A NEW BFS.

x_{31} is the "worst offender" in the sense of the smallest value of t_{ij}. Set $x_{31} = \eta \geq 0$ and perform a PERTURBATION LOOP.

T_1	55	70	35	40
80	$55-\eta$	$25+\eta$		
100		$45-\eta$	35	$20+\eta$
20	η			$20-\eta$

LOOP KEEPS EACH ROW SUM AND EACH COLUMN SUM UNCHANGED.

Feasible solution for $\eta \in [0, 20]$. If $\eta = 20$, x_{34} is pushed out of the basis. This will add $t_{31}\eta = -21(20) = -420$ to z.

T_2	55	70	35	40	u_i
80	$35-\eta$ [13]	$45+\eta$ [11]	[18]	[17]	0
100	η [2]	$25-\eta$ [14]	35 [10]	40 [1]	3
20	20 [5]	[8]	[18]	[11]	-8
v_j	13	11	7	-2	1790 = 2210 - 420

$$t_{ij} = \begin{bmatrix} 0 & 0 & 11 & 19 \\ -14 & 0 & 0 & 0 \\ 0 & 5 & 29 & 21 \end{bmatrix}$$

x_{21} is the worst offender. Perform a perturbation loop to bring x_{21} into the basis. $\eta \in [0, 25]$ is feasible. If $\eta = 25$ we add x_{21} to the basis, pushing out x_{22}.

(**) WORK WITH THE HANDOUT (**)

6.3.3 THE MATRIX METHOD.

The N-W corner method doesn't take into account costs. A method that does is the matrix method, or cheapest cost.

① Find $\min\limits_{i,j} c_{ij}$ and allocate as large as possible x_{ij}

②. Repeat until all supplies empty, all demands met. If a tie, choose arbitrarily between them.

e.g.

T_1	55	70	35	40	u_i
80	13	$50^{+\eta}$ 11	$30^{-\eta}$ 18	17	0
100	$55-\eta$ 2	14	$5^{+\eta}$ 10	40 1	-8
20	η 5	$20^{-\eta}$ 8	18	11	-3
v_j	10	11	18	9	1450

$$t_{ij} = \begin{bmatrix} 3 & 0 & 0 & 8 \\ 0 & 11 & 0 & 0 \\ -2 & 0 & 3 & 5 \end{bmatrix}$$

$\eta = 20$, $z = 1450 - 2(20) = 1410$.

T_2	55	70	35	40	
80		70	10		
100	35		25	40	
20	20				
					1410

which is the optimal solution : see T_4 on the handout.

6.3.4 A NOTE ON DEGENERACY.

Degenerate solutions occur when a bfs has less than $m+n-1$ non-zero elements. It occurs in two ways in the transportation algorithm

1. Initial bfs, supply and demand simultaneously satisfied (and not the final point)
2. There may be a tie in choosing which variable to remove.

In practice, it doesn't present a problem so long as you note which variable $(=0)$ is in the basis and still solve $u_i + v_j = c_{ij}$ for that variable.

EXAMPLE.

$$\underline{c} = \begin{pmatrix} 3 & 9 & 15 & 12 \\ 1 & 5 & 12 & 14 \\ 9 & 4 & 3 & 2 \end{pmatrix} \quad \underline{s} = \begin{pmatrix} 40 \\ 30 \\ 20 \end{pmatrix} \quad \underline{d} = \begin{pmatrix} 50 \\ 20 \\ 10 \\ 10 \end{pmatrix}$$

N-W CORNER:

T_1	50	20	10	10	u_i
40	40 3	9	15	12	0
30	10 1	20 5	12	14	-2
20	9	0 4	10 3	10 2	-3
v_j	3	7	6	5	280

SOLVE $c_{32} = u_3 + v_2$.

$$t_{ij} = \begin{bmatrix} 0 & 2 & 9 & 7 \\ 0 & 0 & 8 & 11 \\ 9 & 0 & 0 & 0 \end{bmatrix}$$

THIS IS THE OPTIMAL SOLUTION!

MATRIX METHOD

T_1	50	20	10	10	u_i
40	$20^{+\eta}$ 3	$20^{-\eta}$ 9	15	12	0
30	$30^{-\eta}$ 1	η 5	12	14	−2
20	9	0 4	10 3	10 2	−5
v_j	3	9	8	7	320

FIVE VARIABLES RATHER THAN SIX. ADD A ZERO!

SOLVING SIMULTANEOUSLY ROW & COLUMN.

$$t_{ij} = \begin{bmatrix} 0 & 0 & 7 & 5 \\ 0 & -2 & 6 & 9 \\ 11 & 0 & 0 & 0 \end{bmatrix}$$

T_2	50	20	10	10	
40	40				
30	10	20			
20		0	10	10	
					280

which is the optimal solution.

Sometimes degenerate solutions involve moving the zero about. e.g. choose x_{13} with that x_{32} as degenerate basic zero.

Move the basic zero from x_{13} to x_{14}.

Example of the transportation algorithm in action

Consider the transportation problem with costs $\mathbf{c}$, supply $\mathbf{s}$ and demand $\mathbf{d}$ given by

$$\mathbf{c} = \begin{pmatrix} 13 & 11 & 18 & 17 \\ 2 & 14 & 10 & 1 \\ 5 & 8 & 18 & 11 \end{pmatrix}, \mathbf{s} = \begin{pmatrix} 80 \\ 100 \\ 20 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 55 \\ 70 \\ 35 \\ 40 \end{pmatrix}.$$

1. Finding an initial BFS

To solve the problem, we first find an initial basic feasible solution using the north-west corner method. We obtain T_1.

T_1	55	70	35	40
80	55	25	0	0
100	0	45	35	20
20	0	0	0	20

2. Checking if BFS is optimal

We now need to check whether this solution is optimal using the asymmetric complementary slackness conditions. For any $x_{ij} > 0$ we require $u_i + v_j = c_{ij}$. We can find the u_i, v_j by working with T_1. We first add as subscripts to each x_{ij} the corresponding cost c_{ij}. As we have one free parameter, we elect to set $u_1 = 0$. We then solve for the remaining u_i, v_j.

T_1	55	70	35	40	u_i
80	$55_{[13]}$	$25_{[11]}$	$0_{[18]}$	$0_{[17]}$	0
100	$0_{[2]}$	$45_{[14]}$	$35_{[10]}$	$20_{[1]}$	3
20	$0_{[5]}$	$0_{[8]}$	$0_{[18]}$	$20_{[11]}$	13
v_j	13	11	7	-2	$z = 2210$

Notice that we have also computed the objective function of the transportation problem

$$z = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} = 13(55) + 11(25) + 14(45) + 10(35) + 1(20) + 11(20) \;=\; 2210.$$

We need to check whether T_1 corresponds to an optimal solution. In particular, is the dual solution feasible? It will be if $u_i + v_j \leq c_{ij}$ for each i, j with $x_{ij} = 0$. We compute $t_{ij} = c_{ij} - u_i - v_j$. For example, $t_{14} = 17 - 0 - (-2) = 19$. Then, the solution will be optimal if $t_{ij} \geq 0$. We obtain

$$[t_{ij}] = \begin{bmatrix} 0 & 0 & 11 & 19 \\ -14 & 0 & 0 & 0 \\ -21 & -16 & -2 & 0 \end{bmatrix} \tag{1}$$

so that the dual solution is not feasible. Our current basic feasible solution is not optimal.

3. Moving to a new BFS

Notice that x_{31} is the "worst offender" in the sense of having the smallest value of t_{ij}. Let's introduce x_{31} into the basis by setting $x_{31} = \eta > 0$. From our initial T_1 we can perform a **perturbation loop** to do so.

T_1	55	70	35	40
80	$55 - \eta$	$25 + \eta$	0	0
100	0	$45 - \eta$	35	$20 + \eta$
20	η	0	0	$20 - \eta$

The solution is feasible for $\eta \in [0, 20]$. If $\eta = 20$ we obtain a new basic feasible solution with x_{31} replacing x_{34} in the basis. We set $\eta = 20$ and recalculate the corresponding values of u_i and v_j giving us the table T_2.

T_2	55	70	35	40	u_i
80	$35_{[13]}$	$45_{[11]}$	$0_{[18]}$	$0_{[17]}$	0
100	$0_{[2]}$	$25_{[14]}$	$35_{[10]}$	$40_{[1]}$	3
20	$20_{[5]}$	$0_{[8]}$	$0_{[18]}$	$0_{[11]}$	-8
v_j	13	11	7	-2	$z = 1790$

Notice that our new value of $z = 1790 = 2210 - 21(20)$ where our choice of $\eta = 20$ and the corresponding $t_{31} = -21$ in (1).

Cycling through the algorithm

We now check if this solution is optimal by once again calculating $t_{ij} = c_{ij} - u_i - v_j$.

$$[t_{ij}] = \begin{bmatrix} 0 & 0 & 11 & 19 \\ -14 & 0 & 0 & 0 \\ 0 & 5 & 19 & 21 \end{bmatrix} \tag{2}$$

The dual solution is once again not feasible. We introduce x_{21} into the basis and perform a perturbation loop.

T_2	55	70	35	40	
80	$35 - \eta$	$45 + \eta$	0	0	
100	η	$25 - \eta$	35	40	
20	20	0	0	0	

The solution is feasible for $\eta \in [0, 25]$. If $\eta = 25$ we obtain a new basic feasible solution with x_{21} replacing x_{22} in the basis. We set $\eta = 25$ and recalculate the corresponding values of u_i and v_j giving us the table T_3, alongside this we compute the corresponding $[t_{ij}]$ and use this to perform, if appropriate a perturbation loop on T_3.

T_3	55	70	35	40	u_i
80	$10_{[13]} - \eta$	$70_{[11]}$	$0_{[18]} + \eta$	$0_{[17]}$	0
100	$25_{[2]} + \eta$	$0_{[14]}$	$35_{[10]} - \eta$	$40_{[1]}$	-11
20	$20_{[5]}$	$0_{[8]}$	$0_{[18]}$	$0_{[11]}$	-8
v_j	13	11	21	12	$z = 1440$

$$[t_{ij}] = \begin{bmatrix} 0 & 0 & -3 & 5 \\ 0 & 14 & 0 & 0 \\ 0 & 5 & 5 & 7 \end{bmatrix}$$

Once again, note that $\eta = 25$ and, from (2), $t_{21} = -14$ and $z = 1440 = 1790 - 14(25)$. We see that T_3 does not represent an optimal solution and we should perform a perturbation loop and, taking $\eta = 10$, introduce x_{13} into the basis for x_{11}. We obtain the table T_4.

T_4	55	70	35	40	u_i
80	$0_{[13]}$	$70_{[11]}$	$10_{[18]}$	$0_{[17]}$	0
100	$35_{[2]}$	$0_{[14]}$	$25_{[10]}$	$40_{[1]}$	-8
20	$20_{[5]}$	$0_{[8]}$	$0_{[18]}$	$0_{[11]}$	-5
v_j	10	11	18	9	$z = 1410$

$$[t_{ij}] = \begin{bmatrix} 3 & 0 & 0 & 8 \\ 0 & 11 & 0 & 0 \\ 0 & 2 & 5 & 7 \end{bmatrix}$$

Note that $\eta = 10$ with $t_{13} = -3$ and $z = 1410 = 1440 - 3(10)$. We see that this solution is optimal as all the $t_{ij} \geq 0$ so the asymmetric complementary slackness conditions are met.