A

30/11/2015.

LECTURE 20: CB1.11.

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} e_1^{(m)} & & & e_2^{(m)} & & & & & e_m^{(m)} & & \\ \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{n}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{n}} & & & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{n}} \end{array} \\
\begin{array}{c} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array} &
\left[\begin{array}{ccccccccccc}
1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \\
1 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 1
\end{array}\right]
\begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \\ x_{21} \\ \vdots \\ x_{2n} \\ \vdots \\ x_{m1} \\ \vdots \\ x_{mn} \end{bmatrix}
\begin{array}{l} \leftarrow \underline{x}_1 \\ \\ \\ \\ \\ \leftarrow \underline{x}_2 \\ \\ \\ \\ \leftarrow \underline{x}_m \end{array}
=
\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \\ d_1 \\ \vdots \\ d_n \end{bmatrix} \\
 & \begin{array}{ccc} \uparrow & & \uparrow \\ e_1^{(n)} & & e_n^{(n)} \end{array}
\end{array}
$$

i.e. $\underline{A}\,\underline{x} = \underline{b}$

NOTE: ① $\underline{A}$ has entries which are either zero or one.
② Each column has only two non-zero entries.
↳ Each row of $\underline{A}^T$ has only two non-zero entries.

Let $\underline{e}_i^{(m)}$ denote the ith column of the $m\times m$ identity matrix
$\underline{e}_j^{(n)}$ denote the jth column of the $n\times n$ identity matrix

then $\underline{A} = (\underline{e}_{11} \cdots \underline{e}_{1n}\ \underline{e}_{21} \cdots \underline{e}_{2n} \cdots \underline{e}_{m1} \cdots \underline{e}_{mn})$ where

$$\underline{e}_{ij} = \begin{pmatrix} \underline{e}_i^{(m)} \\ \underline{e}_j^{(n)} \end{pmatrix}$$

Let $\underline{x} = (\underline{x}_1^T, \ldots, \underline{x}_m^T)^T \in \mathbb{R}^{mn}$ where $\underline{x}_i = (x_{i1}, \ldots, x_{in})^T$
$\underline{c} = (\underline{c}_1^T, \ldots, \underline{c}_m^T)^T \in \mathbb{R}^{mn}$ where $\underline{c}_i = (c_{i1}, \ldots, c_{in})^T$
$\underline{b} = (s_1, \ldots, s_m, d_1, \ldots, d_n)^T \in \mathbb{R}^{m+n}$ then the balanced transportation problem can be written as:

$$\begin{aligned} \text{minimize} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & \underline{A}\,\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0}_{mn} \end{aligned}$$

The dual is:

$$\text{maximise} \quad z' = \underline{b}^T \underline{y}$$
$$\text{subject to} \quad \underline{A}^T \underline{y} \le \underline{c}$$

$$\left[\begin{array}{l} \text{(P) maximise } -\underline{c}^T \underline{x} \text{ subject to } -\underline{A}\,\underline{x} = -\underline{b},\ \underline{x} \ge \underline{0}_{mn} \\ \text{(D) minimise } -\underline{b}^T \underline{y} \text{ subject to } (-\underline{A})^T \underline{y} \ge -\underline{c} \end{array}\right]$$

Let $\underline{y} = (u_1, \dots, u_m, v_1, \dots, v_n)^T = (\underline{u}^T, \underline{v}^T)^T$ then:

$$\underline{A}^T \underline{y} = \begin{bmatrix} \underline{e}_1^{(m)T} & \underline{e}_1^{(n)T} \\ & \vdots \\ \underline{e}_i^{(m)T} & \underline{e}_j^{(n)T} \\ & \vdots \\ \underline{e}_m^{(m)T} & \underline{e}_n^{(n)T} \end{bmatrix} \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \begin{bmatrix} \underline{e}_1^{(m)T} \underline{u} + \underline{e}_1^{(n)T} \underline{v} \\ \vdots \\ \underline{e}_i^{(m)T} \underline{u} + \underline{e}_j^{(n)T} \underline{v} \\ \vdots \\ \underline{e}_m^{(m)T} \underline{u} + \underline{e}_n^{(n)T} \underline{v} \end{bmatrix} \le \begin{bmatrix} c_{11} \\ \vdots \\ c_{ij} \\ \vdots \\ c_{mn} \end{bmatrix}$$

$$\text{i.e. } u_i + v_j \le c_{ij} \quad i = 1, \dots, m,\ j = 1, \dots, n.$$

Similarly, $\underline{b}^T \underline{y} = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j$

That is, the dual is $\quad \text{maximise} \quad z' = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j$

$$\text{subject to} \quad u_i + v_j \le c_{ij} \quad i = 1, \dots, m,\ j = 1, \dots, n.$$

6.2 KEY PROPERTIES OF THE TRANSPORTATION PROBLEM.

THEOREM.

The balanced transportation problem has a finite optimal solution.

Proof.

We shall show both the balanced transportation problem and its dual are both feasible.

Let $x_{ij} = \dfrac{s_i d_j}{\sum_{i=1}^{m} s_i}$ then $\sum_{i=1}^{m} x_{ij} = \dfrac{\left(\sum_{i=1}^{m} s_i\right) d_j}{\sum_{i=1}^{m} s_i} = d_j$

$\sum_{j=1}^{n} x_{ij} = \dfrac{s_i \left(\sum_{j=1}^{n} d_j\right)}{\sum_{i=1}^{m} s_i} = s_i$ as balanced.

$x_{ij} \geq 0$ so btp feasible. For dual, as $c_{ij} \geq 0$ then $u_i = v_j = 0$ so the dual is feasible.

A consequence of weak duality and duality was Corollary 3 which states if (P) and (D) are feasible then $\exists$ finite optimal solutions to (P) and (D) with equal values of their objective functions. $\square$

In the btp, $\sum_{j=1}^{n} x_{ij} = s_i$, $\sum_{i=1}^{n} x_{ij} = d_j$ with $\sum_{j=1}^{n} d_j = \sum_{i=1}^{m} s_i$

so that the $m+n$ constraints are NOT linearly independent.

THEOREM.

Rank $(\underline{A}) = m+n-1$

Proof

See Question Sheet Nine.

Conclusions.

- Using Fundamental Theorem of LP, optimal solution occurs at bfs.
- Rank $(\underline{A}) = m+n-1$. So, bfs has at most $m+n-1$ non-zero values.

6.3 THE TRANSPORTATION ALGORITHM.

We make use of the asymmetric CS theorem. For the btp we have

x is an optimal feasible solution for (P) and y is an optimal feasible solution for (D) if and only if they are feasible solutions and

$$x_{ij}\,[u_i + v_j - c_{ij}] = 0 \quad \forall\, i=1,\dots,m \;\; j=1,\dots,n$$

which is equivalent to

$$x_{ij} > 0 \Rightarrow u_i + v_j = c_{ij} \quad \text{or} \quad u_i + v_j < c_{ij} \Rightarrow x_{ij} = 0.$$

STRATEGY.

① Find a bfs with at most $m+n-1$ non-zero values of x_{ij}
[PRIMAL FEASIBILITY]

② Use asymmetric CS, write down $m+n-1$ equations $u_i + v_j = c_{ij}$. Set (for example) $u_1 = 0$ and solve.
[SATISFYING CS CONDITIONS]

③ If u_i, v_j satisfy $u_i + v_j \le c_{ij}$ $\forall\, i,j$ optimal solution found
[DUAL FEASIBLE].

If not, move to another bfs of btp and repeat.

6.3.1 FINDING AN INITIAL BFS: THE NORTH-WEST CORNER METHOD.

Represent shipments from S_i to D_j as an $m \times n$ matrix.

	d_1	$\dots$	d_n
s_1	x_{11}	$\dots$	x_{1n}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
s_m	x_{m1}	$\dots$	x_{mn}

$$\sum_{j=1}^{n} x_{1j} = s_1 \quad \vdots \quad \sum_{j=1}^{n} x_{mj} = s_m$$

$$\sum_{i=1}^{m} x_{i1} = d_1 \quad \dots \quad \sum_{i=1}^{m} x_{in} = d_n$$