28/9/2015. LECTURE 2: CB1.11. Ⓐ.

2. THE LINEAR PROGRAMMING PROBLEM.

2.1 STANDARD FRAMEWORK.

LP IS A PROBLEM OF MINIMISING OR MAXIMISING A LINEAR FUNCTION GIVEN A SET OF LINEAR EQUALITY OR INEQUALITY PROBLEMS.

EXAMPLE (COST MINIMISATION PROBLEM)

WISH TO FEED CATTLE, MEETING NUTRITIONAL REQUIREMENTS AT MINIMUM COST.
CATTLE MUST RECEIVE MINIMUM QUANTITIES OF CARBOHYDRATE, VITAMINS AND PROTEIN.
TWO FEEDS AVAILABLE: "MOOTASTIC" AND "UDDERLY WONDERFUL"

NUMBER OF UNITS PER KILO OF NUTRIENTS REQUIRED AND COST OF TWO FEEDS.

Food	Carbo	Vit	Pro	Cost	
Mootastic	10	2	2	40	
Udderly Wonderful	5	3	8	80	(per per kilo)
Minimum Requirement.	10	3	4		

LET x_1 BE # KILO OF MOOTASTIC
x_2 BE # KILO OF UDDERLY WONDERFUL

OUR OBJECTIVE IS TO:

(B)

TWO VARIABLES

$$\text{minimise} \quad z = \underbrace{40x_1 + 80x_2}$$
$$\text{subject to} \quad \left.\begin{aligned} 10x_1 + 5x_2 &\geq 10 \\ 2x_1 + 3x_2 &\geq 3 \\ 2x_1 + 8x_2 &\geq 4 \end{aligned}\right\} \text{THREE CONSTRAINTS}$$
$$x_1, x_2 \geq 0$$

PREVIOUS EXAMPLE A MAXIMISATION PROBLEM

BOUND BELOW

THIS EXAMPLE IS A MINIMISATION PROBLEM WITH CONSTRAINTS BOUND BELOW. THIS MORNING'S EXAMPLE WAS A MAXIMISATION PROBLEM WITH CONSTRAINTS BOUND ABOVE.

BOTH PROBLEMS CAN BE EXPRESSED IN THE SAME FRAMEWORK.

DEFINITION.

THE STANDARD MAXIMISATION PROBLEM IS TO FIND A VECTOR $\underline{x} \in \mathbb{R}^n$ THAT WILL

$$\begin{aligned} \text{maximise} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & \underline{A}\,\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{aligned}$$

WHERE THE MATRIX $\underline{A} \in \mathbb{R}^{m\times n}$ WITH $m,n \geq 1$, THE VECTOR $\underline{b} \in \mathbb{R}^m$, $\underline{c} \in \mathbb{R}^n$ AND $\underline{0}_n$ IS THE n-DIMENSIONAL VECTOR OF ZEROS.

$$\underset{n\times 1}{\underline{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}} \qquad \underset{n\times 1}{\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} \qquad \underset{m\times 1}{\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}} \qquad \underline{A} = \overbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}^{n} \Big\} m$$

EXPANDING, WE HAVE:

$$\begin{aligned} \text{maximise} \quad & z = c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} \quad & a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1 \\ & \vdots \\ & a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m \\ & x_j \geq 0 \ \forall\, j = 1, \ldots, n. \end{aligned}$$

DENOTE THE jth COLUMN OF $\underline{A}$ BY $\underline{a}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ SO $\underline{A} = (\underline{a}_1, \ldots, \underline{a}_n)$

THEN, EQUIVALENTLY, WE HAVE

$$\begin{aligned} \text{maximise} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j \underline{a}_j \leq \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{aligned}$$

SUCH A COLUMN REPRESENTATION OF $\underline{A}\,\underline{x}$ WILL BE INSIGHTFUL WHEN WE CONSIDER BASIC AND NON-BASIC SOLUTIONS.

NOTE:

1. $\underline{c}^T \underline{x}$ EQUIVALENT TO $\underline{c} \cdot \underline{x}$ (NOTATION USED IN PAST PAPERS UP TO 2013).

2. $z = z(\underline{x})$ TERMED THE OBJECTIVE FUNCTION.

3. x_j REPRESENTS THE jth VARIABLE $(j = 1, \ldots, n)$

4. $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ IS THE ith CONSTRAINT $(i = 1, \ldots, m)$

5. AT THIS STAGE COULD HAVE $n \leq m$ OR $m \leq n$.
   [LATER EXPLAIN WHEN CONSIDER CANONICAL PROBLEM THAT WE'LL ASSUME $m \leq n$]

(6). ANY $\underline{x} \in \mathbb{R}^n$ SATISFYING BOTH $\underline{A}\,\underline{x} \leq \underline{b}$ AND $\underline{x} \geq \underline{0}_n$ IS A FEASIBLE SOLUTION.

(7) IF $\underline{x}$ IS FEASIBLE AND MAXIMISES z THEN IT'S THE OPTIMAL (FEASIBLE) SOLUTION. IT MAY OR MAY NOT EXIST OR BE UNIQUE.

FOR ANY SET OF VARIABLES $\underline{x}$ WE MAY ASSUME THAT $\underline{x} \geq \underline{0}_n$. IF NOT, WE CAN REWRITE A LINEAR PROGRAMMING PROBLEM TO ENSURE THIS.

(1). IF $x_j \geq l_j$ THEN $x_j - l_j \geq 0$. REPLACE x_j BY $y_j = x_j - l_j$ EVERYWHERE.

(2). IF $x_j \leq u_j$ THEN $u_j - x_j \geq 0$. REPLACE x_j BY $y_j = u_j - x_j$ EVERYWHERE.

(3). IF x_j IS UNCONSTRAINED THEN INTRODUCE w_j, y_j WITH $w_j \geq 0, y_j \geq 0$ AND $x_j = w_j - y_j$

[ANY NUMBER CAN BE EXPRESSED AS THE DIFFERENCE OF TWO NON-NEGATIVE NUMBERS]

DEFINITION.

THE STANDARD MINIMISATION PROBLEM IS TO FIND A VECTOR $\underline{x} \in \mathbb{R}^n$ THAT WILL

$$\begin{aligned} \text{minimise} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & \underline{A}\,\underline{x} \geq \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{aligned}$$

WHERE $\underline{A} \in \mathbb{R}^{m \times n}$, $\underline{b} \in \mathbb{R}^m$, $\underline{c} \in \mathbb{R}^n$, $\underline{0}_n$ ZERO VECTOR.

E.

ANY LP PROBLEM CAN BE EXPRESSED AS A STANDARD MAXIMISATION PROBLEM.

— HERE:

1. MINIMISING $\underline{c}^T \underline{x}$ EQUIVALENT TO MAXIMISING $-\underline{c}^T \underline{x}$

2. CONSTRAINT $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$ EQUIVALENT TO $-\sum_{j=1}^{n} a_{ij} x_j \leq -b_i$

3. CONSTRAINT $\sum_{j=1}^{n} a_{ij} x_j = b_i$ EQUIVALENT TO:

   $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ AND $-\sum_{j=1}^{n} a_{ij} x_j \leq -b_i$

IN THIS COURSE, WE'LL WORK WITH MAXIMISATION.

"STANDARD MAXIMISATION PROBLEM" ≡ "STANDARD PROBLEM".