30/11/2015. LECTURE 19: 3WN2.1.

PROOF.

$\Rightarrow$

Suppose $\underline{x}, \underline{y}$ optimal then $\underline{c}^T \underline{x} = \underline{b}^T \underline{y}$ [Shown results for symmetric dual; results for asymmetric dual identical: see Question Sheet Eight].

$$\underline{y}^T \underline{A} \underline{x} = \underline{y}^T \underline{b} = \underline{c}^T \underline{x}$$

$$\text{i.e. } \underline{x}^T (\underline{A}^T \underline{y} - \underline{c}) = 0 \Rightarrow \sum_{j=1}^{n} \overbrace{x_j}^{\geq 0} \overbrace{((\underline{A}^T \underline{y})_j - c_j)}^{\geq 0} = 0.$$

As $\underline{x}, \underline{y}$ feasible, $x_j \geq 0$ $(\underline{A}^T \underline{y})_j - c_j \geq 0$ so that $x_j ((\underline{A}^T \underline{y})_j - c_j) = 0$ $\forall j = 1, \ldots, n$.

$\Leftarrow$

Suppose $x_j [(\underline{A}^T \underline{y})_j - c_j] = 0$ $\forall j = 1, \ldots, n$ then:

$$\underline{x}^T \underline{A}^T \underline{y} = \underline{c}^T \underline{x}.$$

However, as $\underline{x}$ feasible, $\underline{A} \underline{x} = \underline{b} \Rightarrow \underline{b}^T = \underline{x}^T \underline{A}$ so that $\underline{b}^T \underline{y} = \underline{c}^T \underline{x}$. $\square$

We'll make use of this theorem when studying the transportation algorithm.

6. THE TRANSPORTATION ALGORITHM.

LET $S_1, \ldots, S_m$ denote sources supplying up to $s_1, \ldots, s_m$ units of some commodity which are to be transported to destinations $D_1, \ldots, D_n$ with demands $d_1, \ldots, d_n$. We assume that $s_i, d_j > 0$. Let

x_{ij} be the amount shipped from S_i to D_j
c_{ij} cost per unit of transporting from S_i to D_j

The transportation problem is:

$$\text{minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \quad \text{total cost of shipments}$$

(amount shipped from S_i) subject to $\displaystyle\sum_{j=1}^{n} x_{ij} \leq s_i \qquad i = 1, \ldots, m$

(amount shipped to D_j) $\displaystyle\sum_{i=1}^{m} x_{ij} \geq d_j \qquad j = 1, \ldots, n$

$$x_{ij} \geq 0 \qquad i = 1, \ldots, m, \; j = 1, \ldots, n.$$

NOTES.

① LP problem with mn variables, $m+n$ constraints

② $\underbrace{\sum_{i=1}^{m} s_i}_{\text{total supply}} \geq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \geq \underbrace{\sum_{j=1}^{n} d_j}_{\text{total demand}}$

So, for feasibility, total supply has to be greater than or equal to total demand.

If $\sum_{i=1}^{m} s_i > \sum_{j=1}^{n} d_j$ introduce a dummy destination or "DUMP" D_{n+1} with demand $d_{n+1} = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j$ and cost $c_{i\,n+1} = 0 \;\; \forall\, i = 1, \ldots, m$.

So, objective function unchanged by introduction of D_{n+1} so the two problems have the same optimal solution (solution to original : restrict to x_{ij} for $i = 1, \ldots, m, j = 1, \ldots, n$).

ASSUMPTION.

WLOG we may assume that supply is equal to demand

$$\text{i.e.} \quad \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

In this case, we say that the problem is BALANCED.

In the balanced case, each supply and demand constraint is an equality.

[Otherwise, suppose for example $\sum_{i=1}^{m} x_{ij} > d_j$ then:

$$\sum_{i=1}^{m} s_i \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} > \sum_{j=1}^{n} d_j = \sum_{i=1}^{m} s_i \quad : \text{a contradiction}\;].$$

DEFINITION.

The balanced transportation problem is

$$\text{minimise } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{subject to } \quad \sum_{j=1}^{n} x_{ij} = s_i \qquad i = 1,\dots,m$$

$$\sum_{i=1}^{m} x_{ij} = d_j \qquad j = 1,\dots,n$$

$$x_{ij} \geq 0 \quad i = 1,\dots,m,\; j = 1,\dots,n$$

where $c_{ij} \geq 0$, $s_i > 0$, $d_j > 0$ are given with $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

The main constraints can be written as:

$$\underline{e}_1^{(m)} \qquad \underline{e}_2^{(m)} \qquad \underline{e}_m^{(m)}$$

$$\begin{array}{c} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array}
\left[\begin{array}{ccc|ccc|c|ccc}
\multicolumn{3}{c|}{\overbrace{\hphantom{1\;\cdots\;1}}^{n}} & \multicolumn{3}{c|}{\overbrace{\hphantom{1\;\cdots\;1}}^{n}} & & \multicolumn{3}{c}{\overbrace{\hphantom{1\;\cdots\;1}}^{n}} \\
1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \\
1 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 1
\end{array}\right]
\left[\begin{array}{c} x_{11} \\ \vdots \\ x_{1n} \\ x_{21} \\ \vdots \\ x_{2n} \\ \vdots \\ x_{m1} \\ \vdots \\ x_{mn} \end{array}\right]
=
\left[\begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_m \\ d_1 \\ \vdots \\ d_n \end{array}\right]$$

$$\underline{e}_1^{(n)} \qquad \underline{e}_n^{(n)}$$

$\underline{x}_1$, $\underline{x}_2$, $\underline{x}_n$