23/11/2015. LECTURE 18: CB1.11.

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PROOF.

$\Leftarrow$

If $y_i > 0$ then $(\underline{A}\,\underline{x})_i = b_i$ [i.e. CONSTRAINT IS TIGHT]. Thus,

$$\underline{y}^T \underline{A}\,\underline{x} = \sum_{i=1}^{m} y_i (\underline{A}\,\underline{x})_i = \sum_{i:y_i>0} y_i b_i = \underline{b}^T \underline{y}$$

AS OTHER $y_i = 0$

Similarly, if $x_j > 0$ then $(\underline{A}^T \underline{y})_j = c_j$ [i.e. CONSTRAINT IS TIGHT]. Thus,

$$\underline{x}^T \underline{A}^T \underline{y} = \sum_{j=1}^{n} x_j (\underline{A}^T \underline{y})_j = \sum_{j:x_j>0} x_j c_j = \underline{c}^T \underline{x}$$

AS OTHER $x_j = 0$

Hence, for feasible $\underline{x}, \underline{y}$ we have

$$\underline{b}^T \underline{y} = \underline{y}^T \underline{A}\,\underline{x} = \underline{x}^T \underline{A}^T \underline{y} = \underline{c}^T \underline{x}.$$

Thus, by weak duality corollary 2, $\underline{x}$ is optimal for (P) and $\underline{y}$ optimal for (D).

$\Rightarrow$

Suppose $\underline{x}$ optimal for (P), $\underline{y}$ optimal for (D) then $\underline{c}^T \underline{x} = \underline{b}^T \underline{y}$. Now, from (P), $\underline{A}\,\underline{x} \le \underline{b}$ whilst from (D), $\underline{A}^T \underline{y} \ge \underline{c}$. As $\underline{y} \ge \underline{0}_m$, $\underline{x} \ge \underline{0}_n$ we have:

$$\underline{y}^T \underline{A}\,\underline{x} \le \underline{y}^T \underline{b} = \underline{c}^T \underline{x} \le \underline{x}^T \underline{A}^T \underline{y} = \underline{y}^T \underline{A}\,\underline{x}.$$

i.e. $\underline{y}^T \underline{A}\,\underline{x} = \underline{y}^T \underline{b}$ AND $\underline{x}^T \underline{A}^T \underline{y} = \underline{x}^T \underline{c}$

as $\underline{x}, \underline{y}$ feasible.

$$\Rightarrow \sum_{i=1}^{m} \overbrace{y_i}^{\ge 0} \overbrace{[(\underline{A}\,\underline{x})_i - b_i]}^{\le 0} = 0 \quad \text{AND} \quad \sum_{j=1}^{n} \overbrace{x_j}^{\ge 0} \overbrace{[(\underline{A}^T \underline{y})_j - c_j]}^{\ge 0} = 0$$

i.e. $y_i [(\underline{A}\,\underline{x})_i - b_i] = 0 \quad \forall\ i = 1, \dots, m$

$x_j [(\underline{A}^T \underline{y})_j - c_j] = 0 \quad \forall\ j = 1, \dots, n$

$\square$

EXAMPLE.

$$\begin{array}{lll} \text{maximise} & z = 4x_1 + 2x_2 + 3x_3 & \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \le 12 & (\underline{A}\,\underline{x})_1 \le b_1 \\ & x_1 + 4x_2 + 2x_3 \le 10 & (\underline{A}\,\underline{x})_2 \le b_2 \\ & 3x_1 + x_2 + x_3 \le 10 & (\underline{A}\,\underline{x})_3 \le b_3 \\ & x_1, x_2, x_3 \ge 0. & \end{array}$$

The optimal solution is $x_1 = 2$, $x_2 = 0$, $x_3 = 4$. We can use the complementary slackness conditions to find the optimal solution of the dual problem.

$$\text{NB.} \quad \overset{\underline{A}}{\begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \\ 3 & 1 & 1 \end{pmatrix}} \overset{\underline{x}}{\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}} = \begin{pmatrix} 8 \\ 10 \\ 10 \end{pmatrix} \quad \begin{array}{l} \text{SLACK} \\ \text{TIGHT} \\ \text{TIGHT} \end{array}$$

So, $(\underline{A}\,\underline{x})_1 - b_1 \neq 0$ but $(\underline{A}\,\underline{x})_2 = b_2$

FIRST CONSTRAINT SLACK. $\quad (\underline{A}\,\underline{x})_3 = b_3$

[IN CANONICAL FORM $s_1 \neq 0$, $s_2 = 0$, $s_3 = 0$ OPTIMAL BASIS $\{x_1, x_3, s_1\}$].

If $\underline{x}$ optimal then CS conditions imply $y_1 = 0$ in optimal solution to the dual problem

$$\begin{array}{lll} \text{minimise} & z' = 12y_1 + 10y_2 + 10y_3 & \\ \text{subject to} & 2y_1 + y_2 + 3y_3 \ge 4 & (\underline{A}^T\underline{y})_1 \ge c_1 \\ & 3y_1 + 4y_2 + y_3 \ge 2 & (\underline{A}^T\underline{y})_2 \ge c_2 \\ & y_1 + 2y_2 + y_3 \ge 3 & (\underline{A}^T\underline{y})_3 \ge c_3 \\ & y_1, y_2, y_3 \ge 0 & \end{array}$$

As $x_1 > 0$ and $x_3 > 0$ from the CS conditions,

$$\begin{array}{ll} (\underline{A}^T\underline{y})_1 - c_1 = 0 & 2y_1 + y_2 + 3y_3 = 4 \\ (\underline{A}^T\underline{y})_3 - c_3 = 0 & y_1 + 2y_2 + y_3 = 3 \end{array} \Bigg\} \quad \begin{array}{l} y_2 + 3y_3 = 4 \\ 2y_2 + y_3 = 3 \end{array}$$

$y_1 = 0$

i.e. $y_2 = y_3 = 1$.

We need to check that $y_1=0$, $y_2=1$, $y_3=1$ is feasible (if it wasn't then $\underline{x}$ would not be optimal). Checking the second constraint

$$3(0)+4(1)+1=5\geq 2$$

As $\underline{y}\geq \underline{0}_3$ and $\underline{A}^T\underline{y}\geq \underline{c}$ this is feasible. Thus, it is optimal.

Note: $\overbrace{4(2)+2(0)+3(4)}^{\underline{c}^T\underline{x}} = 20 = \overbrace{12(0)+10(1)+10(1)}^{\underline{b}^T\underline{y}}$.

Symmetric CS conditions.

- If a variable in one problem is positive, corresponding constraint in the other must be tight.
  [Example: $x_1, x_3>0$ so 1^{st}, 3^{rd} constraints of dual tight].
- If a constraint in one problem is not tight, that is it is SLACK, then corresponding variable in the other problem must be zero.
  [Example: 1^{st} constraint of primal slack so $y_1=0$].

As $\underline{x}\geq\underline{0}_n$, $\underline{y}\geq\underline{0}_m$, $\underline{A}\underline{x}\leq\underline{b}$, $\underline{A}^T\underline{y}\geq\underline{c}$ for feasible $\underline{x},\underline{y}$ then CS conditions are equivalent to:

Used these two constraints in the example.

$$\begin{cases} y_i>0 \Rightarrow (\underline{A}\underline{x})_i=b_i \\ \boxed{x_j>0 \Rightarrow (\underline{A}^T\underline{y})_j=c_j} \end{cases} \quad\text{or}\quad \begin{cases} \boxed{(\underline{A}\underline{x})_i<b_i \Rightarrow y_i=0} \\ (\underline{A}^T\underline{y})_j>c_j \Rightarrow x_j=0 \end{cases}$$

[Positivity ⇒ Tight Constraint] [Slackness ⇒ variable zero].

Theorem (Asymmetric Complementary Slackness).
Consider the LP problem in canonical form:

(P) maximise $z=\underline{c}^T\underline{x}$
subject to $\underline{A}\underline{x}=\underline{b}$
$\underline{x}\geq\underline{0}_n$.

(D) minimise $z'=\underline{b}^T\underline{y}$
subject to $\underline{A}^T\underline{y}\geq\underline{c}$

Then $\underline{x} \in \mathbb{R}^n$, $\underline{y} \in \mathbb{R}^m$ optimal for (P) and (D) respectively if and only if they are feasible solutions and

$$x_j \left[(\underline{A}^T \underline{y})_j - c_j \right] = 0 \quad \forall\, j = 1, \ldots, n.$$

[$(\underline{A}\,\underline{x})_i - b_i = 0 \;\; \forall\, i = 1, \ldots, m$ AS CANONICAL SO $\underline{A}\,\underline{x} = \underline{b}$].

which is equivalent to:

$$x_j > 0 \Rightarrow (\underline{A}^T \underline{y})_j = c_j \quad \text{OR} \quad (\underline{A}^T \underline{y})_j > c_j \Rightarrow x_j = 0.$$

→ HERE -

Proof.

$\Rightarrow$

Suppose $\underline{x}, \underline{y}$ optimal then $\underline{c}^T \underline{x} = \underline{b}^T \underline{y}$ [Shown results for symmetric dual; results for asymmetric dual identical: see question sheet eight]. Thus,

$$\underline{y}^T \underline{A}\, \underline{x} = \underline{y}^T \underline{b} = \underline{c}^T \underline{x}$$

i.e. $\underline{x}^T (\underline{A}^T \underline{y} - \underline{c}) = 0 \Rightarrow \sum_{j=1}^{n} x_j \left((\underline{A}^T \underline{y})_j - c_j \right) = 0$

As $\underline{x}, \underline{y}$ feasible $x_j \geq 0$ $(\underline{A}^T \underline{y})_j - c_j \geq 0$ then $x_j \left((\underline{A}^T \underline{y})_j - c_j \right) = 0$ $\forall\, j = 1, \ldots, n$.

$\Leftarrow$

Suppose $x_j \left[(\underline{A}^T \underline{y})_j - c_j \right] = 0 \;\; \forall\, j = 1, \ldots, n$ then:

$$\underline{x}^T \underline{A}^T \underline{y} = \underline{c}^T \underline{x}$$

However, as $\underline{x}$ feasible, $\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{b}^T = \underline{x}^T \underline{A}^T$ so that $\underline{b}^T \underline{y} = \underline{c}^T \underline{x}$. $\square$

We'll make use of this theorem when studying the transportation algorithm.