23/11/2015. LECTURE 17: 3WN2.1.

(A)

PROOF.

As $\underline{x}_0$ is feasible to (P) then $\underline{A}\,\underline{x}_0 \leq \underline{b}$ with $\underline{x}_0 \geq \underline{0}_n$.

As $\underline{y}_0$ is feasible to (D) then $\underline{A}^T \underline{y}_0 \geq \underline{c}$ with $\underline{y}_0 \geq \underline{0}_m$

Premultiplying $\underline{A}\,\underline{x}_0 \leq \underline{b}$ by $\underline{y}_0^T \geq \underline{0}_m^T$ gives

No change in inequality sign as $\underline{y}_0 \geq \underline{0}_m$

$$\underline{y}_0^T \underline{A}\,\underline{x}_0 \leq \underline{y}_0^T \underline{b} = \underline{b}^T \underline{y}_0 \quad (\text{as } \in \mathbb{R}).$$

Similarly, premultiplying $\underline{A}^T \underline{y}_0 \geq \underline{c}$ by $\underline{x}_0^T \geq \underline{0}_n^T$ gives

$$\underline{y}_0^T \underline{A}\,\underline{x}_0 = \underline{x}_0^T \underline{A}^T \underline{y}_0 \geq \underline{x}_0^T \underline{c} = \underline{c}^T \underline{x}_0 \quad (\text{as } \mathbb{R})$$

$$\Rightarrow \quad \underline{c}^T \underline{x}_0 \leq \underline{x}_0^T \underline{A}^T \underline{y}_0 = \underline{y}_0^T \underline{A}\,\underline{x}_0 \leq \underline{b}^T \underline{y}_0 \qquad \square.$$

Let $F_{(P)}$ denote the feasible set of the primal and $F_{(D)}$ the feasible set of the dual. As $\underline{x}_0, \underline{y}_0$ were arbitrary feasible solutions to (P) and (D) respectively,

$$\sup_{\underline{x} \in F_{(P)}} \underline{c}^T \underline{x} \leq \inf_{\underline{y} \in F_{(D)}} \underline{b}^T \underline{y}$$

COROLLARY.

IF the primal problem (P) has a feasible solution but no bounded optimal then (D) has no feasible solution. Likewise, if (D) has a feasible solution but no bounded optimal solution then (P) has no feasible solution.

Proof.

(P) unbounded $\Rightarrow \exists\, \alpha \in \mathbb{R}$ such that $\underline{c}^T \underline{x}_0 > \alpha$ for some $\underline{x}_0 \in F_{(P)}$.

Thus, for ANY $\underline{y}_0 \in F_{(D)}$, $\underline{b}^T \underline{y}_0 > \alpha$ where α is arbitrary.

There's no finite $\underline{y}_0$ with this property (if $\underline{b}^T \underline{y}_0$ finite then $\underline{b}^T \underline{y}_0 = \beta$ so choose $\alpha > \beta$) so no feasible solution to (D).

The alternate argument follows by switching roles. $\square$.

[Essentially, $\infty \le \underline{b}^T \underline{y}_0$; $\underline{c}^T \underline{x}_0 \le -\infty$ no space].

COROLLARY.

Suppose there exist feasible solutions for (P) and (D), say $\underline{x}_0$ and $\underline{y}_0$ respectively, such that $\underline{c}^T \underline{x}_0 = \underline{b}^T \underline{y}_0$. Then $\underline{x}_0$ and $\underline{y}_0$ are optimal for (P) and (D) respectively.

Proof.

For any pair of solutions we have

$$\underline{c}^T \underline{x}_0 \le \sup_{\underline{x} \in F_{(P)}} \underline{c}^T \underline{x} \le \inf_{\underline{y} \in F_{(D)}} \underline{b}^T \underline{y} \le \underline{b}^T \underline{y}_0 .$$

If $\underline{c}^T \underline{x}_0 = \underline{b}^T \underline{y}_0$ then all above inequalities must be equalities so that $\underline{x}_0$ and $\underline{y}_0$ are optimal.

THEOREM (DUALITY THEOREM).

The primal problem (P) has a finite optimal solution IF AND ONLY IF the dual problem (D) does too, in which case the optimal value of their objective functions are the same.

Proof.

Outside the scope of the course.

A very nice observation, useful for the transportation problem, is the following:

Ⓒ

Corollary.

Suppose there exists at least one feasible solution to each of (P) and (D). Then there exist bounded optimal solutions to both (P) and (D) with equal values of the objective functions.

Proof.

Follows from the first corollary and the duality theorem.

3.4 COMPLEMENTARY SLACKNESS.

We look at a further theorem relating primal and dual problems.

Note: $\underline{A}\,\underline{x} \in \mathbb{R}^m$ and $\underline{A}^T \underline{y} \in \mathbb{R}^n$. Let

$$(\underline{A}\,\underline{x}) = ((\underline{A}\,\underline{x})_1, \dots (\underline{A}\,\underline{x})_m)^T$$
$$(\underline{A}^T\underline{y}) = ((\underline{A}^T\underline{y})_1, \dots (\underline{A}^T\underline{y})_n)^T$$

[$(\underline{A}\,\underline{x})_i$ LHS OF ith CONSTRAINT]

[$\underline{A}\,\underline{x} \le \underline{b}$ equivalent to the system $(\underline{A}\,\underline{x})_i \le b_i \quad i = 1, \dots, m$
$\underline{A}^T \underline{y} \ge \underline{c}$ equivalent to the system $(\underline{A}^T\underline{y})_j \ge c_j \quad j = 1, \dots, n$.]

THEOREM [SYMMETRIC COMPLEMENTARY SLACKNESS].

Consider the standard LP problem (P). Then $\underline{x} \in \mathbb{R}^n$ is an optimal feasible solution for (P) and $\underline{y} \in \mathbb{R}^m$ is an optimal solution for (D) if and only if they are feasible solutions and

[$\underline{y}^T \underline{A}\,\underline{x} = \underline{y}^T \underline{b}$ and component wise] $\quad y_i [(\underline{A}\,\underline{x})_i - b_i] = 0 \quad \forall\ i = 1, \dots, m$

[$\underline{x}^T \underline{A}^T \underline{y} = \underline{x}^T \underline{c}$ and component wise]. $\quad x_j [(\underline{A}^T\underline{y})_j - c_j] = 0 \quad \forall\ j = 1, \dots, n$

Proof.

$\Leftarrow$

If $y_i > 0$ then $(\underline{A}\,\underline{x})_i = b_i$ [i.e. CONSTRAINT IS TIGHT].