16|11|2015.

LECTURE 16: OB1.11.

$$x_1 + 2x_2 \le \frac{1}{5}(5x_1 + 20x_2) \le \frac{1}{5}(400) = 80$$

$$x_1 + 2x_2 \le \underbrace{\frac{2}{15}(10x_1 + 15x_2)}_{\frac{4}{3}x_1 + 2x_2;\ \frac{4}{3} \ge 1} \le \frac{2}{15}(450) = 60$$

WE CAN ALSO CONSIDER LINEAR COMBINATIONS OF THE CONSTRAINTS.

x_1 terms: $\frac{5}{15} + \frac{10}{15} = 1$, x_2 terms: $\frac{20}{15} + 1 = \frac{35}{15} \ge 2$

$$x_1 + 2x_2 \le \frac{1}{15}(5x_1 + 20x_2) + \frac{1}{15}(10x_1 + 15x_2)$$

$$\le \frac{1}{15}(400) + \frac{1}{15}(450) = 56\,{}^2/_3.$$

GENERALISING THIS APPROACH, WE HAVE:

$$x_1 + 2x_2 \le y_1(5x_1 + 20x_2) + y_2(10x_1 + 15x_2)$$

$$\le 400y_1 + 450y_2$$

don't change the sign of the inequalities

WHERE WE WANT TO FIND MULTIPLIERS $y_1, y_2 \ge 0$ SUCH THAT

$$5y_1 + 10y_2 \ge 1 \qquad (x_1 \text{ terms})$$

$$20y_1 + 15y_2 \ge 2 \qquad (x_2 \text{ terms})$$

WHICH MAKE THE UPPER BOUND $400y_1 + 450y_2$ AS SMALL AS POSSIBLE.
THAT IS, WE WANT TO:

$$\begin{aligned} \text{minimize} \quad & z' = 400y_1 + 450y_2 && \} \ b^T y. \\ \text{subject to} \quad & 5y_1 + 10y_2 \ge 1 && \multirow{2}{*}{} \\ & 20y_1 + 15y_2 \ge 2 && \} \ A^T y \ge c \\ & y_1, y_2 \ge 0 \end{aligned}$$

THIS IS THE DUAL OF THE INITIAL PROBLEM WHICH WE TERM THE PRIMAL PROBLEM.

NOTE. $y_1 = \frac{1}{25}$ $y_2 = \frac{2}{25}$ GIVES $z' = 52$.

5.2 SYMMETRIC AND ASYMMETRIC DUAL.

DEFINITION.

CONSIDER THE STANDARD MAXIMISATION PROBLEM

$$(P) \qquad \begin{aligned} &\text{maximise} && z = \underline{c}^T \underline{x} \\ &\text{subject to} && \underline{A}\,\underline{x} \le \underline{b} \\ & && \underline{x} \ge \underline{0}_n \end{aligned}$$

WHERE $\underline{c} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^m$, $\underline{A} \in \mathbb{R}^{m \times n}$. ITS SYMMETRIC DUAL IS:

$$(D) \qquad \begin{aligned} &\text{minimise} && z' = \underline{b}^T \underline{y} \\ &\text{subject to} && \underline{A}^T \underline{y} \ge \underline{c} \\ & && \underline{y} \ge \underline{0}_m \end{aligned}$$

NOTES.

1. (P) HAS n VARIABLES, m CONSTRAINTS
   (D) HAS m VARIABLES, n CONSTRAINTS
   [NO ASSUMPTION HERE THAT $m \le n$].
2. ANY LP PROBLEM CAN BE EXPRESSED AS EITHER FORM (P) OR (D).
3. VERY IMPORTANT: DUAL OF THE DUAL IS THE PRIMAL.

THEOREM.

LET (P) BE THE STANDARD MAXIMISATION PROBLEM AND (D) THE SYMMETRIC PRIMAL DUAL OF (P). THEN THE SYMMETRIC DUAL OF (D) IS THE PRIMAL PROBLEM (P).

PROOF.

(D) minimise $z' = \underline{b}^T \underline{y}$
subject to $A^T \underline{y} \geq \underline{c}$
$\underline{y} \geq \underline{0}_m$
(standard minimisation)

Write as $\rightsquigarrow$ a standard maximisation problem.

(D') maximise $z'' = -\underline{b}^T \underline{y}$
subject to $-A^T \underline{y} \leq -\underline{c}$
$\underline{y} \geq \underline{0}_m$

CONSEQUENTLY AS (D') IS A STANDARD MAXIMISATION WE CAN WRITE THE CORRESPONDING DUAL AS

minimise $z''' = -\underline{c}^T \underline{x}$
subject to $(-A)\underline{x} \geq -\underline{b}$
$\underline{x} \geq \underline{0}_n$

Write as a $\rightsquigarrow$ standard maximisation problem

maximise $z = \underline{c}^T \underline{x}$
subject to $A\underline{x} \leq \underline{b}$
$\underline{x} \geq \underline{0}_n$

WHICH WAS THE ORIGINAL PRIMAL (P). $\square$.

DEFINITION.

CONSIDER THE PRIMAL PROBLEM WHICH IS THE CANONICAL MAXIMISATION PROBLEM

$$\begin{aligned} \text{maximise} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{aligned}$$

WHERE $A \in \mathbb{R}^{m \times n}$, $\underline{b} \in \mathbb{R}^m$, $\underline{c} \in \mathbb{R}^n$. ITS ASYMMETRIC DUAL IS THE LP PROBLEM

$$\text{minimize } \quad z' = \underline{b}^T \underline{y}$$
$$\text{subject to } \quad \underline{A}^T \underline{y} \geq \underline{c}$$

NOTICE THAT THIS PROBLEM HAS NO NON-NEGATIVITY REQUIREMENT ON THE ys.

THE MOTIVATION BEHIND THIS DEFINITION IS AS FOLLOWS.

$$\begin{array}{ll} \text{maximize} & z = \underline{c}^T \underline{x} \\ \text{subject to} & \underline{A}\,\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{array} \quad \text{is equivalent to} \quad \begin{array}{ll} \text{maximize} & z = \underline{c}^T \underline{x} \\ \text{subject to} & \underline{A}\,\underline{x} \leq \underline{b} \\ & -\underline{A}\,\underline{x} \leq -\underline{b} \\ & \underline{x} \geq \underline{0}_n \end{array}$$

i.e.
$$\begin{array}{ll} \text{maximize} & z = \underline{c}^T \underline{x} \\ \text{subject to} & \begin{pmatrix} \underline{A} \\ -\underline{A} \end{pmatrix} \underline{x} \leq \begin{pmatrix} \underline{b} \\ -\underline{b} \end{pmatrix} \\ & \underline{x} \geq \underline{0}_n \end{array}$$

$$\tilde{A} = \begin{pmatrix} \underline{A} \\ -\underline{A} \end{pmatrix} \in \mathbb{R}^{2m \times n} \qquad \hat{\underline{b}} = \begin{pmatrix} \underline{b} \\ -\underline{b} \end{pmatrix} \in \mathbb{R}^{2m}$$

WHICH HAS DUAL:

$$(D') \quad \begin{array}{ll} \text{minimize} & z' = (\underline{b}^T \ \ -\underline{b}^T)\, \tilde{\underline{y}} \\ \text{subject to} & (\underline{A}^T \ \ -\underline{A}^T)\, \tilde{\underline{y}} \geq \underline{c} \\ & \tilde{\underline{y}} \geq \underline{0}_{2m} \end{array}$$

LET $\tilde{\underline{y}} = \begin{pmatrix} \underline{v} \\ \underline{v}' \end{pmatrix}$ $\underline{v}, \underline{v}' \in \mathbb{R}^m$. THEN (D') IS EQUIVALENT TO:

$$\begin{array}{ll} \text{minimize} & z' = \underline{b}^T (\underline{v} - \underline{v}') \\ \text{subject to} & \underline{A}^T (\underline{v} - \underline{v}') \geq \underline{c} \\ & \underline{v}, \underline{v}' \geq \underline{0}_m \end{array}$$

LET $\underline{y} = \underline{v} - \underline{v}'$ SO THAT WE HAVE:
$$\begin{array}{ll} \text{minimize} & z' = \underline{b}^T \underline{y} \\ \text{subject to} & \underline{A}^T \underline{y} \geq \underline{c}. \end{array}$$

E.

5.3 THE DUALITY THEOREM.

WE NOW CONSIDER THE RELATIONSHIP BETWEEN SOLUTIONS TO THE DUAL AND PRIMAL.

LET (P) BE THE STANDARD MAXIMISATION PROBLEM AND (D) ITS SYMMETRIC DUAL (i.e. A STANDARD MINIMISATION PROBLEM)

THEOREM (WEAK DUALITY).

SUPPOSE THAT $\underline{x}_0$ IS A FEASIBLE SOLUTION TO (P) AND $\underline{y}_0$ IS FEASIBLE TO (D). THEN:

$$\underline{c}^T \underline{x}_0 \leq \underline{b}^T \underline{y}_0$$

[REALLY WHAT WE DEMONSTRATED IN SECTION 5.1].