16/11/2015. LECTURE 15: 3WN2.1. (A).

** HANDOUT CONTINUED **

TWO-PHASE SIMPLEX METHOD:

PHASE I: SOLVE $\max z' = -\sum_{j=1}^{k} u_j$

subject to $\underline{A}'\underline{x}' = \underline{b}$

$\underline{x}' \geq \underline{0}_{n+k}$

TO FIND EITHER A BFS TO ORIGINAL PROBLEM OR ORIGINAL HAS NO SOLUTION.

PHASE II: SOLVE maximum $z = \underline{c}^T \underline{x}$

subject to $\underline{A}\,\underline{x} = \underline{b}$

$\underline{x} \geq \underline{0}_n$

4.4 SUMMARY OF THE PRIMAL SIMPLEX ALGORITHM

WE CAN NOW SUMMARISE THE IDEAS OF THIS CHAPTER INTO AN ALGORITHM.

STEP 0: PUT YOUR LP PROBLEM INTO CANONICAL FORM
(TYPICALLY $\underline{A}\,\underline{x} = \underline{b}$, $\underline{x} \geq \underline{0}_n$ WITH $\underline{b} \geq \underline{0}_m$)

STEP 1: IF YOU CAN GUESS A FEASIBLE BASIC SOLUTION, USE IT.
(e.g. HAVE $\underline{b} \geq \underline{0}_m$ AND $\underline{A}$ CONTAINS $I_m$ WITHIN ITS COLUMNS)
IF NOT, USE THE TWO-PHASE METHOD AND INTRODUCE ARTIFICIAL VARIABLES.

- IF MAXIMUM $z' = -\sum_{j=1}^{k} u_j < 0$ THEN ORIGINAL LP PROBLEM HAS NO FEASIBLE SOLUTION.
- IF MAXIMUM $z' = 0$ YOU HAVE A STARTING BASIS FOR ORIGINAL PROBLEM.

STEP 2: PERFORM PIVOTS UNTIL EITHER:

- ALL REDUCED COSTS ARE NON-POSITIVE
  (FINAL ROW OF TABLEAU CONTAINS ONLY NON-NEGATIVE VALUES UNDER THE VARIABLES].
  IN WHICH CASE OPTIMAL SOLUTION IS REACHED
  (0 REDUCED COST CORRESPONDS TO NON-UNIQUE SOLUTION).

- THERE IS A COLUMN WITH A POSITIVE REDUCED COST (NEGATIVE ELEMENT IN FINAL ROW) AND NO POSITIVE ELEMENT [COLUMN OF $B^{-1}\underline{N}$ FOR VARIABLE HAS NO POSITIVE VALUES] IN THIS CASE, ORIGINAL PROBLEM IS UNBOUNDED. i.e. NO FINITE OPTIMISING SOLUTION.

NOTE:
IT IS POSSIBLE FOR MANY BASES TO PRODUCE THE SAME BFS (DEGENERATE SOLUTION).
IF YOU'RE NOT CAREFUL, YOU COULD CYCLE THROUGH THESE BASES BUT REFINEMENTS SUCH AS BLAND'S RULE AVOIDS THIS.
[CYCLING ONLY OCCURS IN THE PRESENCE OF DEGENERACY. USE $r_j Q_j$ RULE FOR PIVOTS AND BLAND'S RULE TO BREAK TIES].
IN GENERAL, ALGORITHM SOLVES A LP PROBLEM IN A FINITE NUMBER OF STEPS.

# 5. DUALITY

## 5.1 MOTIVATING EXAMPLE.

CONSIDER OUR ORIGINAL MOTIVATING EXAMPLE.

$$\begin{aligned} \text{maximise} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & 5x_1 + 20x_2 \le 400 \\ & 10x_1 + 15x_2 \le 450 \\ & x_1, x_2 \ge 0 \end{aligned}$$

i.e., $\underline{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\underline{A} = \begin{pmatrix} 5 & 20 \\ 10 & 15 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 400 \\ 450 \end{pmatrix}$

PROBLEM SOLVED WITH $z = 52$ AT $x_1 = 24$, $x_2 = 14$.

SUPPOSE WE CONSIDER USING THE CONSTRAINTS TO BOUND z ABOVE.

HERE — e.g.

$$x_1 + 2x_2 \le \overbrace{\tfrac{1}{5}(5x_1 + 20x_2)}^{x_1 + 4x_2;\ 4 \ge 2} \le \tfrac{1}{5}(400) = 80.$$

$$x_1 + 2x_2 \le \underbrace{\tfrac{2}{15}(10x_1 + 15x_2)}_{\frac{4}{3}x_1 + 2x_2;\ \frac{4}{3} \ge 1} \le \tfrac{2}{15}(450) = 60.$$

WE CAN ALSO CONSIDER LINEAR COMBINATIONS OF THE CONSTRAINTS

$x_1$ terms: $\frac{1}{15}(5) + \frac{1}{15}(10) = 1$   $x_2$ term: $\frac{20}{15} + \frac{15}{15} \ge 2$

$$\begin{aligned} x_1 + 2x_2 &\le \frac{1}{15}(5x_1 + 20x_2) + \frac{1}{15}(10x_1 + 15x_2) \\ &\le \frac{1}{15}(400) + \frac{1}{15}(450) = 56\tfrac{2}{3} \end{aligned}$$

GENERALISING THIS APPROACH, WE HAVE:

$$\begin{aligned} x_1 + 2x_2 &\le y_1(5x_1 + 20x_2) + y_2(10x_1 + 15x_2) \\ &\le 400y_1 + 450y_2 \end{aligned}$$

## Illustrating the two-phase method

**Example 1** *We use the two-phase method to solve the linear programming problem*

$$
\begin{array}{lr}
\textit{maximise} & z = 3x_1 + x_2 \\
\textit{subject to} & x_1 + x_2 + s_1 = 6 \\
 & 4x_1 - x_2 - s_2 = 8 \\
 & 2x_1 + x_2 = 8 \\
 & x_1, x_2, s_1, s_2 \geq 0.
\end{array}
\tag{1}
$$

*In* ***Phase I*** *we solve the auxiliary linear programming problem*

$$
\begin{array}{lrcl}
\textit{maximise} & z' & = & -u_1 - u_2 \\
\textit{subject to} & x_1 + x_2 + s_1 & = & 6 \\
 & 4x_1 - x_2 - s_2 + u_1 & = & 8 \\
 & 2x_1 + x_2 + u_2 & = & 8 \\
 & x_1, x_2, s_1, s_2, u_1, u_2 \geq 0. & &
\end{array}
$$

*to obtain a basic feasible solution to the original problem, (1). In* ***Phase II*** *we solve the original problem.*

*We use $\{s_1, u_1, u_2\}$ as our initial basis with basic feasible solution $s_1 = 6$, $u_1 = 8$ and $u_2 = 8$ with the remaining variables set to zero. To proceed, we want to write our objective function $z'$ as a function of the non-basic variables. To do this note that we have*

$$4x_1 - x_2 - s_2 + u_1 = 8 \tag{2}$$

$$2x_1 + x_2 + u_2 = 8 \tag{3}$$

$$z' + u_1 + u_2 = 0 \tag{4}$$

*so that subtraction of each row containing an artificial variable, from the $z'$ row will eliminate the artificial variables from this equation, that is we perform (4) - (3) - (2) to obtain*

$$z' - 6x_1 + s_2 = -16.$$

*Note that our original objective function $z = 3x_1 + x_2$ does not contain any of the basis variables, $\{s_1, u_1, u_2\}$, so we have $z - 3x_1 - x_2 = 0$. We are ready to commence Phase I.*

### *Phase I*

*We create the initial simplex tableau.*

| | $T_1$ | $z'$ | $z$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $u_1$ | $u_2$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $s_1$ | 0 | 0 | 1 [6] | 1 | 1 | 0 | 0 | 0 | 6 | Initial solution to |
| | $u_1$ | 0 | 0 | 4 [2] | −1 | 0 | −1 | 1 | 0 | 8 | $\underline{A}'\underline{x}' = \underline{b}$ |
| | $u_2$ | 0 | 0 | 2 [4] | 1 | 0 | 0 | 0 | 1 | 8 | |
| Original objective function | *II* | 0 | 1 | −3 | −1 | 0 | 0 | 0 | 0 | 0 | $z - 3x_1 - x_2 = 0$ |
| Auxiliary objective function | *I* | 1 | 0 | −6 | 0 | 0 | 1 | 0 | 0 | −16 | $z' - 6x_1 + s_2 = -16$ |

*We proceed as usual for the simplex method using $z'$ as the objective but performing row operations on the row labelled II which corresponds to the objective function $z$ to ensure that it is always expressed as a function of non-basic variables only. From the bottom row, we see that introducing $x_1$ into the basis will increase $z'$ and $\theta_1 = \min\{6, 8/4, 8/2\} = 2$ so that we push out $u_1$. We perform row operations on $T_1$ to introduce $x_1$ into the basis, obtaining the tableau $T_2$.*

In $T_2$ I can add either $x_2$ (reduced cost 3/2) or $s_2$ (reduced cost 1/2) to the basis. Adding either of these completes Phase I [$u_2$ row regardless of the bottom row] Let's choose freely between them and add $x_2$ to the basis.

don't want to add $u_1$ back into the basis.

| $T_2$ | $z'$ | $z$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | 0 | 0 | 0 | 5/4 | 1 | 1/4 | – | 0 | 4 |
| $x_1$ | 0 | 0 | 1 | −1/4 | 0 | −1/4 | – | 0 | 2 |
| $u_2$ | 0 | 0 | 0 | 3/2 | 0 | 1/2 | – | 1 | 4 |
| $II$ | 0 | 1 | 0 | −7/4 | 0 | −3/4 | – | 0 | 6 |
| $I$ | 1 | 0 | 0 | −3/2 | 0 | −1/2 | – | 0 | −4 |

Maintain $z$ as a function of the non-basic variables

*Note that we do not calculate the values corresponding to $u_1$ as we do not intend to reintroduce this variable. We now remove $u_2$ and can introduce either $x_2$ or $s_2$. We introduce $x_2$ and, performing row operations on $T_2$, obtain the tableau $T_3$.*

| $T_3$ | $z'$ | $z$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | 0 | 0 | 0 | 0 | 1 | −1/6 | – | – | 2/3 |
| $x_1$ | 0 | 0 | 1 | 0 | 0 | −1/6 | – | – | 8/3 |
| $x_2$ | 0 | 0 | 0 | 1 | 0 | 1/3 | – | – | 8/3 |
| $II$ | 0 | 1 | 0 | 0 | 0 | −1/6 | – | – | 32/3 |
| $I$ | 1 | 0 | 0 | 0 | 0 | 0 | – | – | 0 |

$z' = 0$ auxiliary soln.

*We have now moved $u_1$ and $u_2$ out of the basis. We have $z' = 0$ and have a basic feasible solution $(x_1, x_2, s_1, s_2)^T = (8/3, 8/3, 2/3, 0)^T$.*[1]

### *Phase II*

*We can reduce $T_3$ to remove reference to the Phase I variables.*

$z - (c_N^T - c_B^T B^{-1} N) x_N = c_B^T B^{-1} b$

under the basis $\{s_1, x_1, x_2\}$

CHECK SOLUTION

| $T_3$ | $z$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | |
|---|---|---|---|---|---|---|
| $s_1$ | 0 | 0 | 0 | 1 | −1/6 | 2/3 |
| $x_1$ | 0 | 1 | 0 | 0 | −1/6 | 8/3 |
| $x_2$ | 0 | 0 | 1 | 0 | 1/3[8] | 8/3 |
| | 1 | 0 | 0 | 0 | −1/6 | 32/3 |

*Introducing $s_2$ into the basis will improve the objective. We remove $x_2$ from the basis to obtain the tableau $T_4$.*

Note: I would have arrived here one step earlier had I chosen $s_2$ rather than $x_2$ at $T_2$.

| $T_4$ | $z$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | |
|---|---|---|---|---|---|---|
| $s_1$ | 0 | 0 | 1/2 | 1 | 0 | 2 |
| $x_1$ | 0 | 1 | 1/2 | 0 | 0 | 4 |
| $s_2$ | 0 | 0 | 3 | 0 | 1 | 8 |
| | 1 | 0 | 1/2 | 0 | 0 | 12 |

*Reading the bottom row we have $z = 12 - \frac{1}{2}x_2$. The basic feasible solution $(x_1, x_2, s_1, s_2)^T = (4, 0, 2, 8)^T$ is optimal with $z = 12$.*

[1]It can be readily checked that we have not made a mistake in our algebra by checking that (1) is satisfied at this solution: $\frac{8}{3} + \frac{8}{3} + \frac{2}{3} = 6$, $4\frac{8}{3} - \frac{8}{3} = 8$ and $2\frac{8}{3} + \frac{8}{3} = 8$. Also $z = 3\frac{8}{3} + \frac{8}{3} = \frac{32}{3}$ which verifies our final equation.