9/11/2015. LECTURE 14: CBI.11. (B)

WLOG ASSUME k EQUATIONS ARE THE FIRST k ROWS OF $\underline{A}$. DENOTE THE ARTIFICIAL VARIABLES BY $\underline{u}^T = (u_1, \dots, u_k)$ AND THE SLACK VARIABLES IN THE FINAL $m-k$ ROWS BY $\underline{\xi}$.

LET $I_m = (\underline{e}_1, \dots, \underline{e}_m)$ SO $\underline{e}_j$ REPRESENTS THE jth COLUMN OF THE $m \times m$ IDENTITY MATRIX. THE ADDITION OF ARTIFICIAL VARIABLES GIVES:

$$\underline{A}\,\underline{x} + \sum_{j=1}^{k} u_j \underline{e}_j = \underline{b} \Rightarrow \underline{A}'\,\underline{x}' = \underline{b}$$

WHERE $\underline{A}' = (\underline{A} \mid \underline{e}_1 \dots \underline{e}_k)$ [ESSENTIALLY ADDED TO $\underline{A}$ THE MISSING COLUMNS OF $\underline{I}_m$]

AND $\underline{x}'^T = (x_1, \dots, x_n, u_1, \dots, u_k)$

THESE INCLUDE THE SLACK VARIABLES: $m-k$ FROM TYPE ① CONSTRAINTS AND AT MOST k FROM TYPE ② CONSTRAINTS.

NOTE: Rank $(\underline{A}')$ = Rank $(\underline{A})$. IDEA OF PHASE I IS TO SOLVE AN AUXILIARY LP PROBLEM

$$\text{maximise } z' = -\sum_{j=1}^{k} u_j \qquad (*)$$

(NOTE $z' \le 0$ AND $z' = 0$ CORRESPONDS TO $u_i = 0$ $i = 1, \dots, k$).

$$\text{subject to } \underline{A}'\,\underline{x}' = \underline{b}$$
$$\underline{x}' \ge \underline{0}_{n+k}$$

USING THE INITIAL BASIS $\{\underline{u}^T, \underline{\xi}^T\}$. BY DESIGN THIS CORRESPONDS TO AN INITIAL BFS $\{\underline{u}^T, \underline{\xi}^T\} = \underline{b} \ge \underline{0}_m$.

NOTE: $\underline{x}' \ge \underline{0}_{n+k}$ MEANS THAT $\underline{u} \ge \underline{0}_k$. ANY SOLUTION TO (*) THAT ACHIEVES $z' = 0$ MUST HAVE $\underline{u} = \underline{0}_k$.

AS THIS CAN BE FOUND AT A BFS TO (*) THEN CAN FIND A BFS TO (*)

WITH THE u_i'S ALL NON-BASIC (THUS, SET TO ZERO), THE CORRESPONDING SOLUTION WILL BE A BFS OF $\underline{A}\underline{x} = \underline{b}$ (AND SO HAVE $\underline{x} \geq \underline{0}_n$).

[NB. ANY SOLUTION TO $\underline{A}\underline{x} = \underline{b}$, $\underline{x} \geq \underline{0}_n$ SATISFIES $\underline{A}'\underline{x}' = \underline{b}$, $\underline{x}' \geq \underline{0}_{n+k}$

WITH $\underline{x}' = (\underline{x}^T, \underline{0}^T)^T$ i.e. ARTIFICIAL VARIABLES SET TO ZERO]

HENCE, IF OPTIMAL SOLUTION TO z' DOES NOT HAVE $z' = 0$

i.e. NO SOLUTION WITH $u_1 = \cdots = u_k = 0$

THEN THERE IS NO FEASIBLE SOLUTION TO THE ORIGINAL LP PROBLEM.

4.3.2 PHASE II.

ASSUMING PHASE I IS SUCCESSFUL AND WE HAVE AN INITIAL BFS, WE USE THIS AS STARTING POINT FOR SOLVING ORIGINAL PROBLEM.

EXAMPLE: ILLUSTRATING THE TWO-PHASE METHOD:

SEE THE HANDOUT.

INITIAL BFS FOR PHASE I. TO OPERATE THE SIMPLEX ALGORITHM ON PHASE I, EXPRESS z' IN TERMS OF THE NON-BASIC VARIABLES.

THIS CAN BE EASILY FOUND FROM ADDING THE ROWS OF $\underline{A}'$ WHICH CONTAIN AN ARTIFICIAL VARIABLE.

EXAMPLE.

$$4x_1 - x_2 - s_2 + u_1 = 8$$
$$2x_1 + x_2 + u_2 = 8$$
$$\Rightarrow \quad 6x_1 - s_2 + u_1 + u_2 = 16$$

i.e. $z' = -u_1 - u_2 = -16 + 6x_1 - s_2$.

CREATE THE SIMPLEX TABLEAU FOR PHASE I.

(c).

INCLUDE THE OBJECTIVE FUNCTION FOR PHASE II TO ENSURE IN CORRECT FORM WHEN WE FIND AN INITIAL BFS TO IT.

HANDOUT

Illustrating the two-phase method

Example 1 *We use the two-phase method to solve the linear programming problem*

$$\underline{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 4 & -1 & 0 & -1 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{lr} \text{maximise} & z = 3x_1 + x_2 \\ \text{subject to} & x_1 + x_2 + s_1 = 6 \\ & 4x_1 - x_2 - s_2 = 8 \\ & 2x_1 + x_2 = 8 \\ & x_1, x_2, s_1, s_2 \geq 0. \end{array} \tag{1}$$

$\underline{A}$ does not have the second and third column of $\underline{I}_3$.

*In **Phase I** we solve the auxiliary linear programming problem*

$$\underline{A}' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 4 & -1 & 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{lrcl} \text{maximise} & z' & = & -u_1 - u_2 \\ \text{subject to} & x_1 + x_2 + s_1 & = & 6 \\ & 4x_1 - x_2 - s_2 + u_1 & = & 8 \\ & 2x_1 + x_2 + u_2 & = & 8 \\ & x_1, x_2, s_1, s_2, u_1, u_2 \geq 0. \end{array}$$

$\underline{A}'$ now includes $\underline{I}_3$ within its columns.

$\underline{x}' = (x_1, x_2, s_1, s_2, u_1, u_2)^T$

*to obtain a basic feasible solution to the original problem, (1). In **Phase II** we solve the original problem.*

We use $\{s_1, u_1, u_2\}$ as our initial basis with basic feasible solution $s_1 = 6$, $u_1 = 8$ and $u_2 = 8$ with the remaining variables set to zero. To proceed, we want to write our objective function z' as a function of the non-basic variables. To do this note that we have

$$4x_1 - x_2 - s_2 + u_1 = 8 \tag{2}$$

$$2x_1 + x_2 + u_2 = 8 \tag{3}$$

$$z' + u_1 + u_2 = 0 \tag{4}$$

so that subtraction of each row containing an artificial variable, from the z' row will eliminate the artificial variables from this equation, that is we perform (4) - (3) - (2) to obtain

$$z' - 6x_1 + s_2 = -16.$$

Note that our original objective function $z = 3x_1 + x_2$ does not contain any of the basis variables, $\{s_1, u_1, u_2\}$, so we have $z - 3x_1 - x_2 = 0$. We are ready to commence Phase I.

Phase I

We create the initial simplex tableau.

	T_1	z'	z	x_1		x_2	s_1	s_2	u_1	u_2		
	s_1	0	0	1	[6]	1	1	0	0	0	6	Initial solution to
	u_1	0	0	4	[2]	−1	0	−1	1	0	8	$\underline{A}'\underline{x}' = \underline{b}$
	u_2	0	0	2	[4]	1	0	0	0	1	8	
Original objective function	II	0	1	−3		−1	0	0	0	0	0	$z - 3x_1 - x_2 = 0$
Auxiliary object function	I	1	0	−6		0	0	1	0	0	−16	$z' - 6x_1 + s_2 = -16$

We proceed as usual for the simplex method using z' as the objective but performing row operations on the row labelled II which corresponds to the objective function z to ensure that it is always expressed as a function of non-basic variables only. From the bottom row, we see that introducing x_1 into the basis will increase z' and $\theta_1 = \min\{6, 8/4, 8/2\} = 2$ so that we push out u_1. We perform row operations on T_1 to introduce x_1 into the basis, obtaining the tableau T_2.

IN T_2, I CAN ADD EITHER x_2 OR s_2 TO THE BASIS AS THE CORRESPONDING REDUCED COSTS IN z' ARE POSITIVE. ADDING EITHER OF THESE COMPLETES PHASE I [u_2 ROW NEGATIVE OF BOTTOM ROW]. WE CAN CHOOSE FREELY BETWEEN THEM. LET'S CHOOSE x_2.

DON'T WANT TO ADD u_1 BACK INTO THE BASIS.

T_2	z'	z	x_1	x_2	s_1	s_2	u_1	u_2	
s_1	0	0	0	5/4	1	1/4	–	0	4
x_1	0	0	1	−1/4	0	−1/4	–	0	2
u_2	0	0	0	3/2	0	1/2	–	1	4
II	0	1	0	−7/4	0	−3/4	–	0	6
I	1	0	0	−3/2	0	−1/2	–	0	−4

EXPRESS z AS A FUNCTION OF THE NON-BASIC VARIABLE

Note that we do not calculate the values corresponding to u_1 as we do not intend to reintroduce this variable. We now remove u_2 and can introduce either x_2 or s_2. We introduce x_2 and, performing row operations on T_2, obtain the tableau T_3.

T_3	z'	z	x_1	x_2	s_1	s_2	u_1	u_2	
s_1	0	0	0	0	1	−1/6	–	–	2/3
x_1	0	0	1	0	0	−1/6	–	–	8/3
x_2	0	0	0	1	0	1/3	–	–	8/3
II	0	1	0	0	0	−1/6	–	–	32/3
I	1	0	0	0	0	0	–	–	0

ACHIEVED $z'=0$.

AUXILIARY SOLVED!

We have now moved u_1 and u_2 out of the basis. We have $z' = 0$ and have a basic feasible solution $(x_1, x_2, s_1, s_2)^T = (8/3, 8/3, 2/3, 0)^T$[1] KEY POINT: CAN CHECK THIS!

Phase II

We can reduce T_3 to remove reference to the Phase I variables.

$z-(\underline{c}_N^T-\underline{c}_B^T\underline{B}^{-1}\underline{N})\underline{x}_N=\underline{c}_B^T\underline{B}^{-1}\underline{b}$ UNDER THE BASIS $\{s_1, x_1, x_2\}$.

T_3	z	x_1	x_2	s_1	s_2	
s_1	0	0	0	1	−1/6	2/3
x_1	0	1	0	0	−1/6	8/3
x_2	0	0	1	0	1/3 [8]	8/3
	1	0	0	0	−1/6	32/3

$\underline{A}\underline{x}=\underline{b}$ REPRESENTED AS $\underline{x}_B+\underline{B}^{-1}\underline{N}\underline{x}_N=\underline{B}^{-1}\underline{b}$ UNDER BASIS $\{s_1, x_1, x_2\}$.

Introducing s_2 into the basis will improve the objective. We remove x_2 from the basis to obtain the tableau T_4.

NOTE I WOULD HAVE ARRIVED HERE ONE STEP EARLIER HAD I CHOSEN s_2 RATHER THAN x_2 AT T_2.

T_4	z	x_1	x_2	s_1	s_2	
s_1	0	0	1/2	1	0	2
x_1	0	1	1/2	0	0	4
s_2	0	0	3	0	1	8
	1	0	1/2	0	0	12

Reading the bottom row we have $z = 12 - \frac{1}{2}x_2$. The basic feasible solution $(x_1, x_2, s_1, s_2)^T = (4, 0, 2, 8)^T$ is optimal with $z = 12$.

[1] It can be readily checked that we have not made a mistake in our algebra by checking that (1) is satisfied at this solution: $\frac{8}{3} + \frac{8}{3} + \frac{2}{3} = 6$, $4\frac{8}{3} - \frac{8}{3} = 8$ and $2\frac{8}{3} + \frac{8}{3} = 8$. Also $z = 3\frac{8}{3} + \frac{8}{3} = \frac{32}{3}$ which verifies our final equation.