9/11/2015.

(A).

LECTURE 13: 3WN2.1.

4.3 TWO-PHASE METHOD; ARTIFICIAL VARIABLES.

IN OUR EXAMPLES SO FAR WE HAVE STARTED THE SIMPLEX ALGORITHM BY LETTING THE SLACK VARIABLES FORM A BASIS (AS $\underline{b} \geq \underline{0}_m$) IT IS NOT ALWAYS STRAIGHTFORWARD TO FIND AN INITIAL BFS.

EXAMPLE

EXPRESSED SO THAT $\underline{b} \geq \underline{0}_3$

maximise $z = 3x_1 + x_2$
subject to
$$x_1 + x_2 \leq 6$$
$$4x_1 - x_2 \geq 8$$
$$2x_1 + x_2 = 8$$
$$x_1, x_2 \geq 0$$

CANONICAL FORM $\rightsquigarrow$

maximise $z = 3x_1 + x_2$
subject to
$$x_1 + x_2 + s_1 = 6$$
$$4x_1 - x_2 - s_2 = 8$$
$$2x_1 + x_2 = 8$$
$$x_1, x_2, s_1, s_2 \geq 0$$

THERE IS NO IMMEDIATE BFS. WE SOLVE THIS BY USING ARTIFICIAL VARIABLES AND THE TWO-PHASE METHOD.

4.3.1 PHASE I.

THERE ARE THREE POSSIBLE CONSTRAINTS FOR A LP PROBLEM.

① $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ WHERE $b_i \geq 0$

ADD SLACK VARIABLE $s_i \geq 0 \Rightarrow \sum_{j=1}^{n} a_{ij} x_j + s_i = b_i$

NOTE. $s_i = b_i \geq 0$ IS A SOLUTION TO THIS EQUATION.
ADD s_i TO THE INITIAL BASIS.

② $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ WHERE $b_i < 0$

ADD SLACK VARIABLE $s_i \geq 0$

(B)

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i \Rightarrow -\sum_{j=1}^{n} a_{ij} x_j - s_i = -b_i \quad (>0)$$

NO IMMEDIATE NON-NEGATIVE (i.e. FEASIBLE) SOLUTION.
ADD IN ARTIFICIAL VARIABLE $u_i \geq 0$

$$-\sum_{j=1}^{n} a_{ij} x_j - s_i + u_i = -b_i$$

NOTE: $u_i = -b_i > 0$ IS A SOLUTION TO THIS EQUATION

ADD u_i TO THE INITIAL BASIS.

③ $\sum_{j=1}^{n} a_{ij} x_j = b_i$ WHERE $b_i \geq 0$ (CAN ALWAYS ENSURE AN EQUALITY IS EXPRESSED THIS WAY).

ADD IN AN ARTIFICIAL VARIABLE $u_i \geq 0$

$$\sum_{j=1}^{n} a_{ij} x_j + u_i = b_i$$

NOTE: $u_i = b_i \geq 0$ IS A SOLUTION TO THIS EQUATION.

ADD u_i TO THE INITIAL BASIS.

EXAMPLE (REVISITED).

OUR CONSTRAINTS BECOME

$x_1 + x_2 \leq 6$	$x_1 + x_2 + s_1 = 6$ (TYPE ①)
$4x_1 - x_2 \geq 8$	$4x_1 - x_2 - s_2 + u_1 = 8$ (TYPE ②)
$2x_1 + x_2 = 8$	$2x_1 + x_2 + u_2 = 8$ (TYPE ③)

AN IMMEDIATE BFS TO THIS MODIFIED SYSTEM IS

$$s_1 = 6, \ u_1 = 8, \ u_2 = 8.$$

THUS, WE HAVE SLACK VARIABLES IN TYPE (1) AND (2) CONSTRAINTS
ARTIFICIAL VARIABLES IN TYPE (2) AND (3) CONSTRAINTS.

SUPPOSE WE WISH TO SOLVE

$$\begin{aligned} \text{maximise} \quad & z = \underline{c}^T \underline{x} \\ \text{subject to} \quad & \underline{A}\,\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0}_n \end{aligned}$$

WHERE $\text{Rank}(\underline{A}) = m$ AND WE HAVE WRITTEN $\underline{A}$ TO ENSURE $\underline{b} \geq \underline{0}_m$. SUPPOSE OUR SYSTEM CONTAINS:

- $k \leq m$ EQUATIONS WHOSE ORIGINAL FORM, THAT IS BEFORE THE ADDITION OF ANY SLACK VARIABLES, WAS OF THE TYPE $\underbrace{\sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ WHERE } b_i < 0}_{\text{TYPE (2)}}$ OR $\underbrace{\sum_{j=1}^{n} a_{ij} x_j = b_i}_{\text{TYPE (3)}}$ AND TO WHICH WE WILL ADD AN ARTIFICIAL VARIABLE.
  THIS WILL GIVE US k BASIC VARIABLES.

- $m-k$ EQUATIONS WHOSE ORIGINAL FORM WAS $\underbrace{\sum_{j=1}^{n} a_{ij} x_j \leq b_i}_{\text{TYPE (1)}}$ WHERE $b_i \geq 0$. ADD THE CORRESPONDING SLACK VARIABLE TO THE BASIS.
  THIS WILL GIVE US A FURTHER $m-k$ BASIC VARIABLES.

WLOG ASSUME k EQUATIONS ARE THE FIRST k ROWS OF $\underline{A}$. DENOTE THE ARTIFICIAL VARIABLES BY $\underline{u}^T = (u_1, \dots, u_k)$ AND THE SLACK VARIABLES IN THE FINAL $m-k$ ROWS BY $\underline{s}$.

HERE.

LET $I_m = (\underline{e}_1, \dots, \underline{e}_m)$ SO $\underline{e}_j$ REPRESENTS jth COLUMN OF THE $m \times m$ IDENTITY MATRIX.