2|11|2015.

LECTURE 11: 3WN2.1.

LAST TIME:

	T_1	z	x_1	x_2	s_1	s_2	
EQ 1	s_1	0	5	20	1	0	400
EQ 2	s_2	0	10	15	0	1	450
EQ 3		1	-1	-2	0	0	0

ADD x_2 TO THE BASIS, PUSHING OUT s_1

SWITCH

T_2	z	x_1	x_2	s_1	s_2	
x_2	0	1/4	1	1/20	0	20
s_2	0	25/4	0	-3/4	1	150
	1	-1/2	0	1/10	0	40

[MULTIPLY EQ1 BY 1/20; SUBTRACT 15/20 OF EQ1 FROM EQ2; ADD 1/10 OF EQ1 TO EQ3]

NOTE THAT $z = 40 + \frac{1}{2}x_1 - \frac{1}{10}s_1$. CURRENT BASIS CORRESPONDS TO $z = 40$ WHICH COULD BE INCREASED BY ADDING x_1 TO THE BASIS.

STEP THREE: CHANGING THE BASIS, INTRODUCING x_1.

s_1 WILL REMAIN NON-BASIC AND SO EQUAL TO 0.

$\frac{1}{4}x_1 + x_2 = 20$ SO $x_2 = 0 \Rightarrow x_1 = 80$

$\frac{25}{4}x_1 + s_2 = 150$ $s_2 = 0 \Rightarrow x_1 = 24$.

[TAKE $x_1 = M \geq 0$ THEN $(M, 20 - \frac{1}{4}M, 0, 150 - \frac{25}{4}M)$ SOLVES $\underline{A}\underline{x} = \underline{b}$. CHOOSE M TO GET A BFS].

i. ADD x_1, REMOVE x_2.

$x_1 = 80$, $s_2 = -350$ NOT FEASIBLE.

BASIC SOLUTION $(80, 0, 0, -350)^T$

ii. ADD x_1, REMOVE s_2

$x_1 = 24$, $x_2 = 14$

BASIC SOLUTION $(24, 14, 0, 0)^T$ FEASIBLE. PERFORM ROW OPERATIONS ON T_2 TO OBTAIN T_3.

- MULTIPLY EQ2 BY $4/25$
- SUBTRACT $1/25$ EQ2 FROM EQ1
- ADD $2/25$ EQ2 TO EQ3.

T_3	z	x_1	x_2	s_1	s_2	
x_2	0	0	1	$2/25$	$-1/25$	14
x_1	0	1	0	$-3/25$	$4/25$	24
	1	0	0	$1/25$	$2/25$	52

$\underline{x}^* = (24, 14, 0, 0)^T$.

$$z = 52 - \frac{1}{25} s_1 - \frac{2}{25} s_2 \leq 52 = z(\underline{x}^*)$$

ALL OF THE REDUCED COSTS ARE ≤ 0. THIS IS THE OPTIMAL SOLUTION (UNIQUE AS ALL REDUCED COSTS < 0).

* SEE PICTURE*

4.2 MOVING TO A BETTER BASIS.

(WLOG CURRENT BASIS SOLUTION). CONSIDER BASIS SOLUTION

$\underline{x}_B = (x_1, \dots, x_m)^T$, $\underline{x}_N = (x_{m+j}, \dots, x_n)^T$ $\underline{B} = (\underline{a}_1, \dots, \underline{a}_m)$

ASSUME BASIC SOLUTION IS FEASIBLE SO $\underline{B}^{-1}\underline{b} \geq \underline{0}_m$.

BASIC FEASIBLE SOLUTION HAS $x_{m+j}=0 \quad \forall j=1,\dots,n-m$.
AND, FOR ANY SOLUTION TO $\underline{A}\,\underline{x}=\underline{b}$,

$$\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b} \qquad (*)$$

LET α_{ij} DENOTE (i,j)th entry OF $\underline{B}^{-1}\underline{N}$ AND β_i iTH ELEMENT OF $\underline{B}^{-1}\underline{b}$. THEN $(*)$ IS EQUIVALENT TO:

T	z	m BASIS VARIABLES	n−m NON-BASIS	
BASIS VARIABLES	0 ⋮ 0	I_m	$\underline{B}^{-1}\underline{N}$	$\underline{B}^{-1}\underline{b}$
	1	$\underline{0}_m^T$	$-(\underline{c}_N^T-\underline{c}_B^T\underline{B}^{-1}\underline{N})$	$\underline{c}_B^T\underline{B}^{-1}\underline{b}$

$$\begin{aligned}
x_1 + \alpha_{11}x_{m+1} + \alpha_{12}x_{m+2} + \cdots + \alpha_{1\,n-m}x_n &= \beta_1 \\
x_2 + \alpha_{21}x_{m+1} + \alpha_{22}x_{m+2} + \cdots + \alpha_{2\,n-m}x_n &= \beta_2 \qquad (**) \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
x_m + \alpha_{m1}x_{m+1} + \alpha_{m2}x_{m+2} + \cdots + \alpha_{m\,n-m}x_n &= \beta_m
\end{aligned}$$

CURRENT BASIS SOLUTION HAS $x_{m+j}=0 \quad \forall j=1,\dots,n-m$. SUPPOSE INTRODUCE x_{m+j} INTO THE BASIS BUT ALL OTHER NON-BASIC VARIABLES REMAIN EQUAL TO ZERO. THEN $(**)$ BECOMES:

$$x_i + \alpha_{ij}x_{m+j} = \beta_i \qquad i=1,\dots,m$$

$$\left[\ \underline{x}_B + x_{m+j}(\underline{B}^{-1}\underline{N})_j = \underline{B}^{-1}\underline{b}\ \right].$$

- IF $\alpha_{ij}>0$ THEN INCREASING x_{m+j} (FROM 0) DECREASES x_i
- IF $\alpha_{ij}=0$ THEN INCREASING x_{m+j} HAS NO EFFECT ON x_i
- IF $\alpha_{ij}<0$ THEN INCREASING x_{m+j} INCREASES x_i

HERE.

WE CAN ONLY PUSH OUT AN x_i FOR WHICH $\alpha_{ij}>0$ ‖ WE REDUCE x_i TO ZERO (i.e. NON-BASIC) WHEN:

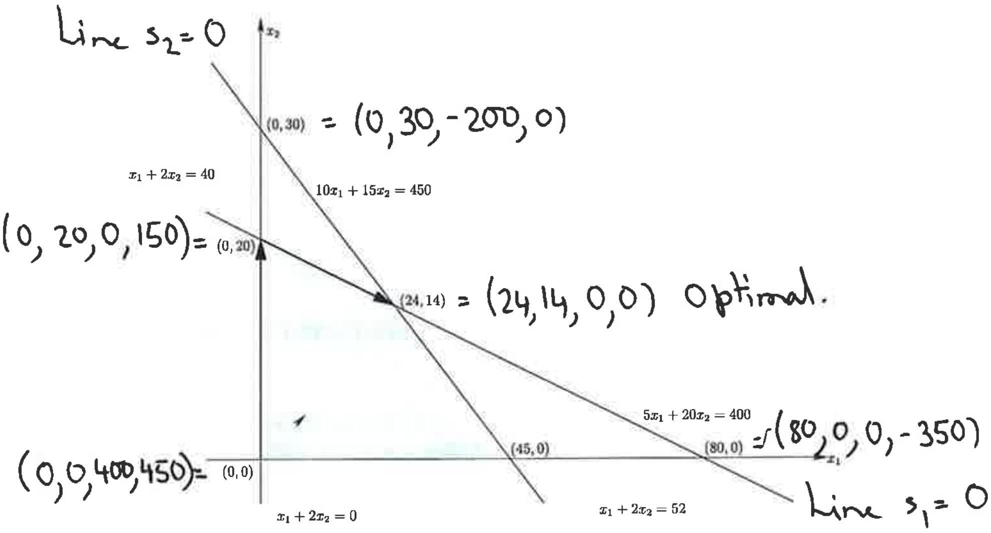

Figure 1: Representation of the simplex algorithm for the original standard problem. Step One corresponds to the extreme point $(0,0)$. In Step Two we introduce x_2 into the basis which increases the objective function. Pushing out s_1 corresponds to moving to $(0,20)$ which is feasible and the move we make; pushing out s_2 corresponds to moving to $(0,30)$ which is not feasible. In Step Three we introduce x_1 into the basis. Pushing out s_2 corresponds to moving to $(24,14)$ which is feasible and optimal and the move we make; pushing out x_2 corresponds to moving to $(80,0)$ which is not feasible.

Step Two: Changing the basis, introducing x_2.

T_1	z	x_1	x_2	s_1	s_2	
s_1	0	5	20	1	0	400
s_2	0	10	15	0	1	450
	1	-1	-2	0	0	0

$\rightsquigarrow$

T_2	z	x_1	x_2	s_1	s_2	
x_2	0	$1/4$	1	$1/20$	0	20
s_2	0	$25/4$	0	$-3/4$	1	150
	1	$-1/2$	0	$1/10$	0	40

- To maintain feasibility, replace s_1 by x_2 in the basis
- Basic feasible solution is $(0,20,0,150)$ with $z = 40 + \frac{1}{2}x_1 - \frac{1}{10}s_1$.
- There is a positive reduced cost: introducing x_1 into the basis will increase z.

Step Three: Changing the basis, introducing x_1.

T_2	z	x_1	x_2	s_1	s_2	
x_2	0	$1/4$	1	$1/20$	0	20
s_2	0	$25/4$	0	$-3/4$	1	150
	1	$-1/2$	0	$1/10$	0	40

$\rightsquigarrow$

T_3	z	x_1	x_2	s_1	s_2	
x_2	0	0	1	$2/25$	$-1/25$	14
x_1	0	1	0	$-3/25$	$4/25$	24
	1	0	0	$1/25$	$2/25$	52

- To maintain feasibility, replace s_2 by x_1 in the basis
- Basic feasible solution is $(24,14,0,0)$ with $z = 52 - \frac{1}{25}s_1 - \frac{2}{25}s_2$.
- All the reduced costs are negative: this is the optimal solution.