26/10/2015. LECTURE 10: CB1.11. A.

KEY EQUATIONS: $$\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b}$$

$$z(\underline{x}) = \underline{c}_B^T \underline{B}^{-1}\underline{b} + \underbrace{(\underline{c}_N^T - \underline{c}_B^T \underline{B}^{-1}\underline{N})}_{\underline{r}^T}\underline{x}_N$$

BFS $\underline{x}^* = \begin{pmatrix} \underline{B}^{-1}\underline{b} \\ \underline{0}_{n-m} \end{pmatrix} (\geq 0)$ $\quad z(\underline{x}) = z(\underline{x}^*) + \sum_{j=1}^{n-m} r_j x_{m+j}$

$\Rightarrow$ $\underline{x}^*$ OPTIMAL IF $r_j \leq 0 \; \forall j = 1, \ldots, n-m$. IF NOT, INTRODUCE x_{m+j} WITH $r_j > 0$ INTO THE BASIS.

EXAMPLE (MOTIVATING EXAMPLE FROM LECTURE 1; QUESTION 3 OF QS2).

$\Rightarrow$ SLIDE.

STEP ONE: INITIAL BFS.

WE NEED TO FIND AN INITIAL BFS. THIS CAN BE TRICKY: WE'LL EXAMINE THIS FURTHER WHEN CONSIDERING ARTIFICIAL VARIABLES.

IN THIS CASE, IMMEDIATE THAT THE BASIS CONSISTING OF THE SLACK VARIABLES GIVES A SOLUTION $\underline{x}^* = (0 \;\; 0 \;\; 400 \;\; 450)$.
THIS CORRESPONDS TO $\underline{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{B}^{-1}$ AND THE ORIGINAL SYSTEM OF EQUATIONS IS IN THE FORM

$$\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b}$$

UNDER THIS BASIS, $\underline{c}_B^T = (0 \;\; 0)$ $\quad \underline{c}_N^T = (1 \;\; 2)$ SO

$$z = \underline{c}_B^T \underline{B}^{-1}\underline{b} + (\underline{c}_N^T - \underline{c}_B^T \underline{B}^{-1}\underline{N})\underline{x}_N$$
$$= 0 + x_1 + 2x_2.$$ [REDUCED COSTS: $\underline{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$]

B.

WE EXPRESS THESE TWO EQUATIONS IN A TABLEAU, T_1.

VARIABLES

	T_1	z	x_1	x_2	s_1	s_2	
EQ 1	s_1	0	5	20	1	0	400
EQ 2	s_2	0	10	15	0	1	450
EQ 3		1	-1	-2	0	0	0

BASIS

$\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N$

CURRENT VALUE OF z UNDER BFS

BASIC VARIABLES HAVE ZERO ENTRIES IN THIS EQUATION

$$z - (\underline{c}_N^T - \underline{c}_B^T \underline{B}^{-1}\underline{N})\underline{x}_N = \underline{c}_B^T \underline{B}^{-1}\underline{b}$$

THE SOLUTION IS NOT OPTIMAL AS THE REDUCED COSTS ARE ALL POSITIVE SO ANY SOLUTION WITH x_1, x_2 NOT BOTH EQUAL TO ZERO WILL INCREASE z.

i.e. INTRODUCING EITHER x_1 OR x_2 INTO THE BASIS WILL INCREASE z (PROVIDING SOLUTION IS NOT DEGENERATE!).

LET'S CHOOSE TO INTRODUCE x_2 INTO THE BASIS. (CHANGING A SINGLE VARIABLE IN THE BASIS WILL BE "STRAIGHTFORWARD")

STEP TWO: CHANGING THE BASIS, INTRODUCING x_2.

WE HAVE TO DECIDE WHICH VARIABLE TO REMOVE FROM THE BASIS TO INCLUDE x_2. [MAINTAIN THE BASIS TO BE OF SIZE $m=2$]

x_1 WILL REMAIN EQUAL TO ZERO (AS REMAINS NON-BASIC), SO SOLUTIONS TO $\underline{A}\,\underline{x} = \underline{b}$ OF THIS FORM HAVE:

$$20x_2 + s_1 = 400$$
$$15x_2 + s_2 = 450$$

- REMOVE s_1 FROM THE BASIS i.e. SET $s_1 = 0$.

$$\Rightarrow \quad x_2 = \frac{400}{20} = 20.$$

$$s_2 = 450 - 15\left(\frac{400}{20}\right) = 150.$$

BASIC FEASIBLE SOLUTION $(0, 20, 0, 150)^T$.

• REMOVE s_2 FROM THE BASIS i.e. SET $s_2 = 0$.

$$\Rightarrow \quad x_2 = \frac{450}{15} = 30.$$

$$s_1 = 400 - 20(30) = -200.$$

BASIC SOLUTION IS $(0, 30, -200, 0)^T$

REMOVE s_1 FROM THE BASIS ("PUSH-OUT") TO MAINTAIN FEASIBILITY. NOTE: $20 = \min\{20, 30\}$

OUR NEW BASIS IS $\{x_2, s_2\}$ WITH BFS $(0, 20, 0, 150)^T$. WE WANT TO WRITE $\underline{A}\,\underline{x} = \underline{b}$ AND z AS THE BASIS DECOMPOSITIONS

$$\underline{x}_B + \underline{B}^{-1}\underline{N}\,\underline{x}_N = \underline{B}^{-1}\underline{b} \qquad (*)$$

$$z = \underline{c}_B^T\,\underline{B}^{-1}\underline{b} + (\underline{c}_N^T - \underline{c}_B^T\,\underline{B}^{-1}\underline{N})\underline{x}_N \qquad (**)$$

WE COULD COMPUTE THESE DIRECTLY OR PERFORM ROW OPERATIONS ON T_1 TO OBTAIN T_2 WHICH HAS THIS REPRESENTATION.

• WE WANT A "1" IN THE FIRST EQUATION OF COLUMN x_2 AND A ZERO EVERYWHERE ELSE.

- MULTIPLY EQ 1 BY $1/20$
- SUBTRACT $15/20$ OF EQ 1 FROM EQ 2
- ADD $2/20$ OF EQ 1 TO EQ 3

WE OBTAIN THE TABLEAU T_2 REPRESENTING (*), (**) UNDER THE BASIS $\{x_2, s_2\}$.

NOTE s_1 PUSHED OUT. THIS COLUMN REPRESENTS ROW OPERATIONS PERFORMED.

HERE: FIRST EQUATION COMPLETE.

T_2	z	x_1	x_2	s_1	s_2	
x_2	0	$1/4$	1	$1/20$	0	20
s_2	0	$25/4$	0	$-3/4$	1	150
	1	$-1/2$	0	$1/10$	0	40

NOTE THAT $z = 40 + \frac{1}{2}x_1 - \frac{1}{10}s_1$. CURRENT BASIS CORRESPONDS TO $z=40$ WHICH COULD INCREASE BY ADDING x_1 TO THE BASIS.

STEP THREE: CHANGING THE BASIS, INTRODUCING x_1.

s_1 STILL NON-BASIC SO KEEP s_1 EQUAL TO ZERO.

$$\frac{1}{4}x_1 + x_2 = 20$$

$$\frac{25}{4}x_1 + s_2 = 150$$

ADD x_1, REMOVE x_2 $\Rightarrow$ $x_1 = \frac{20}{1/4} = 80$

$$s_2 = 150 - \frac{25}{4}(80) = -350.$$

BASIC SOLUTION $(80, 0, 0, -350)^T$ IS NOT FEASIBLE.

ADD x_1, REMOVE s_2 $\Rightarrow$ $x_1 = \frac{150}{25/4} = 24$

$$x_2 = 20 - \frac{1}{4}(24) = 14$$