28/9/2015. LECTURE 1: 3WN 2.1. (A).

1. INTRODUCTION.

1.1 WHAT IS OPERATIONAL RESEARCH?

- OFTEN CONCERNED WITH DETERMINING THE MAXIMUM PROFIT / PERFORMANCE / YIELD OR MINIMUM LOSS / RISK / COST OF SOME REAL-WORLD OBJECTIVE.

- ANALYSIS OF RESOURCE ALLOCATION PROBLEMS MILITARY, BUSINESS, MANUFACTURING.

  e.g. TRANSPORTATION PROBLEM.
  TESCO HAS OVER 20 UK DISTRIBUTION CENTRES SUPPLYING OVER 2,900 STORES WITH OVER 40,000 DIFFERENT PRODUCTS.
  HOW SHOULD IT SUPPLY THE STORES FROM THE DISTRIBUTION CENTRES TO MINIMISE TRANSPORTATION COSTS?

1.2 MOTIVATING EXAMPLE OF LINEAR PROGRAMMING.

IN THIS COURSE, FOCUS UPON LINEAR PROGRAMMING IN WHICH WE SEEK TO:

MINIMISE OR MAXIMISE A LINEAR FUNCTION GIVEN A SET OF LINEAR EQUALITY OR INEQUALITY CONSTRAINTS.

** EXAMPLE **

THE EXAMPLE IS ONE OF PROFIT MAXIMISATION.

Example 1 *A furniture company plans to make two products, oak chairs and oak tables, from its available resources. It has 400 board feet of oak available and 450 hours of labouring time. A chair takes 10 hours to produce, utilising 5 board feet whilst each table takes 15 hours to produce requiring 20 board feet. Both items are sold at a profit, tables earning twice the profit of chairs. Let x_1 denote the number of chairs produced and x_2 the number of tables. Our aim is to*

OBJECTIVE FUNCTION →

$$\begin{aligned} \textit{maximise} \quad z &= x_1 + 2x_2 \leftarrow \text{PROFIT} \\ \textit{subject to} \quad 5x_1 + 20x_2 &\leq 400 \leftarrow \text{CONSTRAINT ON OAK} \\ 10x_1 + 15x_2 &\leq 450 \leftarrow \text{CONSTRAINT ON TIME} \\ x_1, x_2 &\geq 0 \leftarrow \text{WILL SHOW THAT NON-NEGATIVITY} \end{aligned}$$

OF VARIABLES IS A USUAL REQUIREMENT

This is a linear programming problem with two variables and two constraints.[1] *As this is a problem in $\mathbb{R}^2$ we can solve the problem graphically.*

[1] Strictly there are four constraints but, as we shall shortly see, non-negativity of variables is a usual requirement.

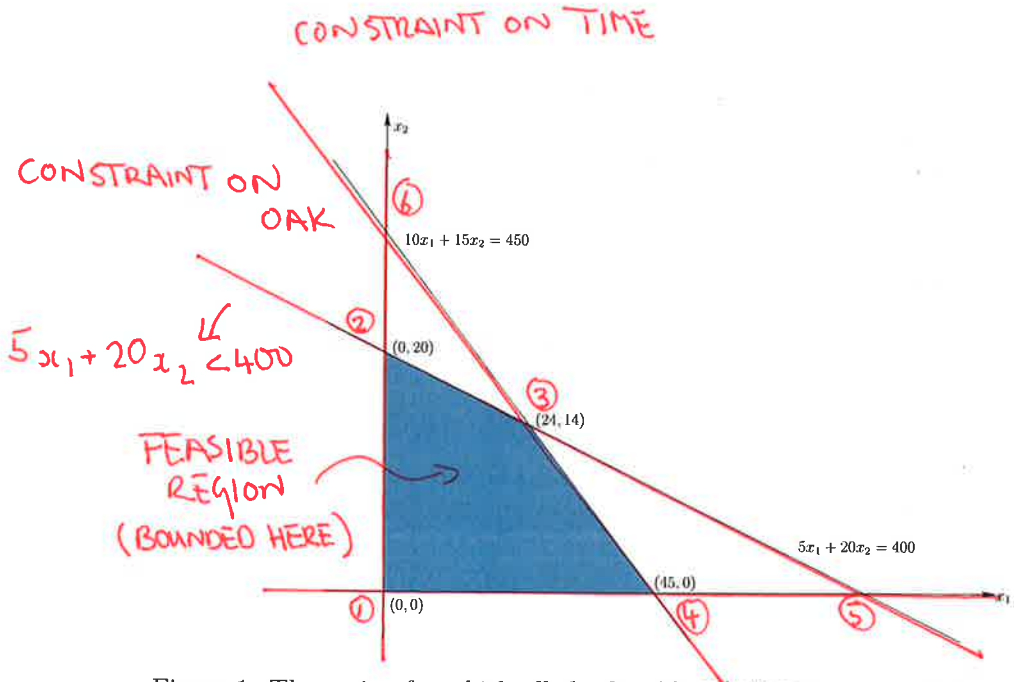
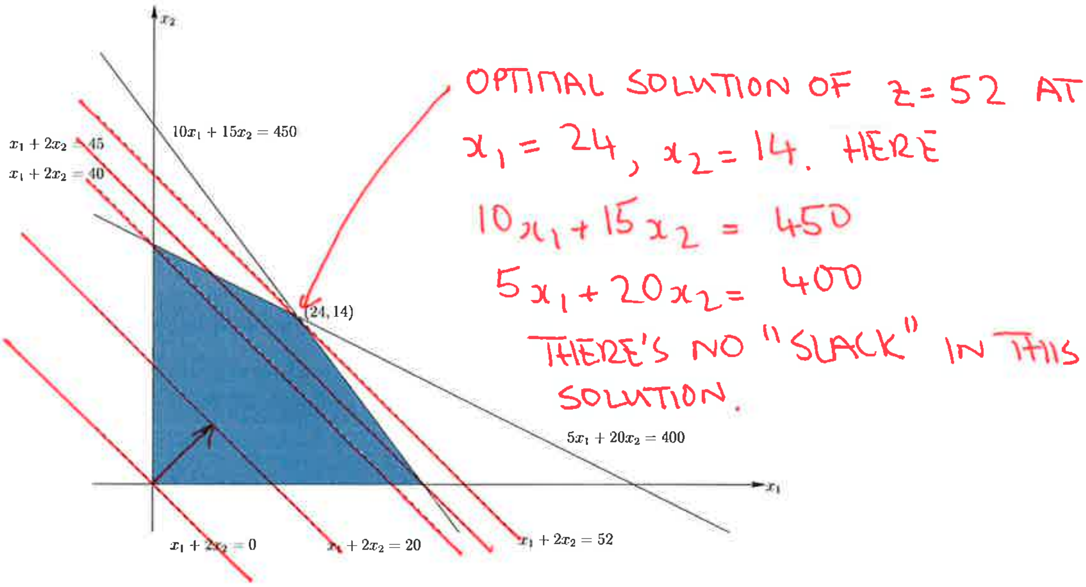


Figure 1: The region for which all the feasible solutions to Example 1 must lie. The region has four extreme points: $(0, 0)$, $(0, 20)$, $(24, 14)$ and $(45, 0)$.

In Figure 1 we illustrate the ***feasible region****, that is the set of points with coordinates* (x_1, x_2) *that satisfy all of the constraints.*

FEASIBLE REGION HAS FOUR EXTREME POINTS WHICH OCCUR AT THE INTERSECT OF CONSTRAINTS.



OPTIMAL SOLUTION OF $z = 52$ AT $x_1 = 24$, $x_2 = 14$. HERE
$10x_1 + 15x_2 = 450$
$5x_1 + 20x_2 = 400$
THERE'S NO "SLACK" IN THIS SOLUTION.

Figure 2: Translating the objective function of Example 1 through the feasible region. The maximum of 52 occurs at the extreme point $(24, 14)$.

The objective is to maximise the linear function $z = x_1 + 2x_2$. Figure 2 illustrates how we do this. If we fix $z = 0$ we can see that the objective function passes through the origin. Increasing the value of z corresponds to translating the objective function through the feasible region (in a northeast direction). For example, Figure 2 shows the objective function for increasing values of z. Note that when $z = 40$ we obtain the maximum feasible value of x_2 and when $z = 45$ we obtain the maximum feasible value of x_1. Continuing to increase z we see that ***the maximum is achieved at the extreme point*** $(24, 14)$ *where $z = 52$.*

NOTES.

1. SOLUTION TO EXAMPLE OCCURRED AT AN EXTREME POINT.
THIS IS NOT A SURPRISE.

WE'LL LATER SHOW, IN THE FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING, THAT FOR A LP PROBLEM IF AN OPTIMAL SOLUTION EXISTS THEN THE OPTIMAL SOLUTION CAN BE FOUND AT AN EXTREME POINT. e.g. IN THIS CASE, THE FEASIBLE REGION IS BOUNDED SO AN OPTIMAL SOLUTION EXISTS.

2. ASSUMED THAT BOTH x_1 AND x_2 ARE NOT INTEGERS. IN THIS COURSE (IN GENERAL), WE'LL ASSUME VARIABLES ARE (NON-NEGATIVE) REALS.

INTEGER PROGRAMMING PROBLEM: NP-HARD.
LINEAR PROGRAMMING PROBLEM CAN BE SOLVED IN POLYNOMIAL TIME.

THE COURSE DIVIDES INTO TWO PARTS:

1. LINEAR PROGRAMMING PROBLEM.
   - GENERIC METHOD, SIMPLEX METHOD, FOR SOLVING
   - EXAMINE THE CONCEPT OF DUALITY.

2. MORE COMPLICATED CLASS OF LP PROBLEMS
   e.g. TRANSPORTATION PROBLEM.

   METHODS OF SOLUTION TO THESE ARE VARIANTS OF THE SIMPLEX METHOD.

WE'LL BE INTERESTED IN BOTH THE "HOW" AND "THE WHY"
i.e. WE WON'T JUST PRESENT ALGORITHMS BUT WILL DEMONSTRATE WHY THEY WORK.